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AUTOMORPHIC FORMS OF $\bar\partial$-COHOMOLOGY TYPE AS COHERENT COHOMOLOGY CLASSES

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The arithmetic theory of holomorphic automorphic forms is most naturally treated in terms of a certain family of vector bundles on Shimura varieties, called *automorphic vector bundles.* Let $M = \Gamma\backslash X$ be a connected component of a Shimura variety; here X is the Hermitian symmetric space of noncompact type associated to the semisimple Lie group G and Γ is an arithmetic subgroup of G. We write $X = G/K_\infty$; let σ be a finite-dimensional representation of K_∞; then σ determines an automorphic vector bundle $[\mathscr{V}_\sigma]$ over M (cf. §2 for definitions). We view $[\mathscr{V}_\sigma]$ as a locally free coherent sheaf over the quasiprojective variety M.

When M is compact, its cohomology with coefficients in $[\mathscr{V}_\sigma]$ can be computed by applying Hodge theory to the Dolbeault complex of $[\mathscr{V}_\sigma]$, endowed with a G-invariant metric. This situation has been studied by a number of authors, notably by Schmid [42]. The harmonic $(0, q)$-forms with values in $[\mathscr{V}_\sigma]$ correspond to the occurrence in $L_2(\Gamma\backslash G)$ of a certain class of unitary representations of G: namely, those with $\bar\partial$-cohomology with coefficients in σ, defined as in §4 below. Here the $\bar\partial$-cohomology of the representation π with coefficients in σ is defined to be the relative Lie algebra cohomology group $H^*(\mathfrak{P}, K_\infty, \pi_0 \otimes \sigma)$, where $\mathfrak{P}$ is a subalgebra of $\mathrm{Lie}(G)_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$ containing $\mathrm{Lie}(K_\infty)_{\mathbb{C}}$ and π_0 is the $(\mathfrak{g}, K_\infty)$ module associated to π.

The unitary representations with $\bar\partial$-cohomology have yet to be classified. They include the discrete series and, more generally, any representation of G with $(\mathfrak{g}, K)$-cohomology (in the sense of [12] and [53]), but there are others as well which play an important role in applications, as we explain below (cf. §4).

It is known [25] that automorphic vector bundles have models over number fields which are compatible with the canonical models [47], [34] of Shimura varieties; thus their sections rational over a given number field k define k-arithmetic (holomorphic) automorphic forms. Shimura has

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introduced a number of tests for the arithmeticity of a holomorphic automorphic form, based on its properties as a holomorphic function on a Hermitian symmetric domain [48], [49]. It has been shown that these tests are compatible with the above definition of rationality, which derives from algebraic geometry. It is natural to study as well the arithmetic of the higher cohomology of automorphic vector bundles. As was explained in [26], this is in fact essential if one hopes to find relations between special values of automorphic L-functions and periods of integrals, along the lines of Deligne's conjecture [18].

We are led naturally to ask two questions:

I. To what extent can higher cohomology of automorphic vector bundles be represented by automorphic forms in the noncompact case?

II. How can we define a notion of arithmeticity for nonholomorphic automorphic forms of $\bar{\partial}$-cohomology type?

The method proposed in this article to deal with question I is based on Mumford's theory of toroidal compactifications of Shimura varieties [2]. Let (G, X) be the basic datum defining the Shimura variety $M = M(G, X)$; here G is a reductive group over $\mathbb{Q}$ and X is a finite union of Hermitian symmetric domains, homogeneous under $G(\mathbb{R})$. Let π be a unitary representation of $G(\mathbb{R})$ and π_0 its associated $(\mathfrak{g}, K_\infty)$-module, where $\mathfrak{g} = \mathrm{Lie}(G)$ and K_∞ is the subgroup of $G(\mathbb{R})$ stabilizing some point in X. Let σ and $[\mathscr{V}_\sigma]$ be as above, and assume π has $\bar{\partial}$-cohomology in degree q with coefficients in σ. Denote by $\mathscr{A}_0(G)$ the space of cusp forms on $G(\mathbb{Q})\backslash G(\mathbf{A})$. A *cusp form of type* π is an element of $\mathscr{A}_0(G)$ lying in $\mathrm{Im}\,\varphi$ for some $\varphi \in \mathrm{Hom}_{(\mathfrak{g}, K_\infty)}(\pi_0, \mathscr{A}_0(G))$.

By analogy with the case of compact Shimura varieties, one would expect elements of $\mathrm{Hom}_{(\mathfrak{g}, K_\infty)}(\pi_0, \mathscr{A}_0(G))$ to define cohomology classes of M in degree q with coefficients in $[\mathscr{V}_\sigma]$. This is not true in general, but there is an adequate substitute. Let $K \subset G(\mathbf{A}^f)$ be an open compact subgroup, and let ${}_KM$ be the corresponding Shimura variety at finite level. Given any toroidal compactification ${}_K\bar{M}$ of ${}_KM$, there are two canonical extensions of $[\mathscr{V}_\sigma]$ to vector bundles over ${}_K\bar{M}$, denoted $[\mathscr{V}_\sigma]^{\mathrm{can}}$ and $[\mathscr{V}_\sigma]^{\mathrm{sub}}$ and a homomorphism $[\mathscr{V}_\sigma]^{\mathrm{sub}} \to [\mathscr{V}_\sigma]^{\mathrm{can}}$ of coherent sheaves. We prove the following results, which together go a long way toward answering question I:

A. *The cohomology groups* $H^*({}_K\bar{M}, [\mathscr{V}_\sigma]^{\mathrm{can}})$ *and* $H^*({}_K\bar{M}, [\mathscr{V}_\sigma]^{\mathrm{sub}})$ *are independent of the choice of toroidal compactification, and the spaces*

$$\tilde{H}^q([\mathscr{V}_\sigma]^{\mathrm{can}}) = \varinjlim_K H^q({}_K\bar{M}, [\mathscr{V}_\sigma]^{\mathrm{can}}),$$

$$\tilde{H}^q([\mathscr{V}_\sigma]^{\mathrm{sub}}) = \varinjlim_K H^q({}_K\bar{M}, [\mathscr{V}_\sigma]^{\mathrm{sub}})$$

are admissible $G(\mathbf{A}^f)$ *modules for any* q (Propositions (2.4) and (2.6)), *with natural rational structures over the field of definition* L_σ *of* $[\mathscr{V}_\sigma]$ (Proposition (2.8)).

We note that L_σ is always a number field which can be explicitly determined in terms of the representation σ [25].

B. *Let* $\bar{H}^q([\mathscr{V}_\sigma])$ *denote the image of* $\tilde{H}^q([\mathscr{V}_\sigma]^{\mathrm{sub}})$ *in* $\tilde{H}^q([\mathscr{V}_\sigma]^{\mathrm{can}})$. *For each* q, *there is a natural imbedding of* $G(\mathbf{A}^f)$ *modules*:

$$\text{(0.1)} \qquad \mathrm{Hom}_{(\mathfrak{g}, K_\infty)}(\pi_0, \mathscr{A}_0(G)) \otimes H^q(\mathfrak{P}, K_\infty, \pi_0 \otimes \sigma) \hookrightarrow \bar{H}^q([\mathscr{V}_\sigma])$$

(Proposition (3.6)).

C. *More generally, let* $\mathscr{A}_{(2)}(G)$ *denote the space of all square integrable automorphic forms on* $G(\mathbb{Q})\backslash G(\mathbf{A})$, *in the sense defined in* §5. *Then there is a natural homomorphism of* $G(\mathbf{A}^f)$*-modules*

$$\text{(0.2)} \quad \bigoplus_\pi \mathrm{Hom}_{(\mathfrak{g}, K_\infty)}(\pi_0, \mathscr{A}_{(2)}(G)) \otimes H^q(\mathfrak{P}, K_\infty, \pi_0 \otimes \sigma) \to \tilde{H}^q([\mathscr{V}_\sigma]^{\mathrm{can}})$$

whose image contains $\bar{H}^q([\mathscr{V}_\sigma])$, *where* π *runs through the set of all unitary representations of* $G(\mathbb{R})$ (Theorem (5.3)).

D. *If* π *is a discrete series representation whose Harish-Chandra parameter is sufficiently "far from the walls", and* σ *is irreducible, then the homomorphism in* (0.1) *is an isomorphism for all* q. *Moreover, under these hypotheses,* $H^q(\mathfrak{P}, K_\infty, \pi_0 \otimes \sigma) = \{0\}$ *except in one dimension* $q = q_\pi$, *and* $\dim H^{q_\pi}(\mathfrak{P}, K_\infty, \pi_0 \otimes \sigma) = 1$ (Corollary (5.3.3)).

The canonical extension $[\mathscr{V}_\sigma]^{\mathrm{can}}$ was introduced by Mumford in [36]. The idea of studying higher cohomology with coefficients in $[\mathscr{V}_\sigma]^{\mathrm{can}}$ was first suggested by Faltings [20], [21] who has since demonstrated their importance for the study of the Hodge structures attached to Siegel modular forms [22]. The fact that cusp forms of type π actually define cohomology classes is based on the Dolbeault complex with logarithmic singularities introduced in [28] and generalized in §3, and makes use of ideas due to Borel [8].

An attempt to provide some motivation for question II can be found in [26], which also describes the idea behind the criteria for arithmeticity studied in the present article. Roughly speaking, for a given representation π with $\bar\partial$-cohomology in degree q, the idea is to find a Hermitian symmetric subdomain $X^{\#}$ of dimension q, and to study the restrictions of automorphic forms of type π to $X^{\#}$ and its translates by elements of $G(\mathbb{Q})$. Because of the condition on the dimension of $X^{\#}$, Serre duality permits us to relate these restrictions to holomorphic automorphic forms on $X^{\#}$, whose arithmetic properties are already well understood.

The technique sketched above works only when the restriction of π to the automorphism group $G^{\#}(\mathbb{R})$ of $X^{\#}$ contains discrete series representations $\pi^{\#}$ of $G^{\#}(\mathbb{R})$ as *direct factors*. Using a suggestion of R. Howe, we show (Theorem (7.4)) that this technique is effective provided $\pi^{\#}$ is of the *integrable* discrete series. This is a technical assumption which can presumably be lifted with more work. Whenever the image of the homomorphism (0.1) is an L_σ-rational subspace, e.g., in the cases described in D above, this technique then provides necessary and sufficient conditions for a cusp form on $G(\mathbb{Q})\backslash G(\mathbf{A})$ of type π to be rational over L_σ (Theorem (7.6)).

The rationality criteria introduced in §7 are worked out explicitly in §8 for coherent cohomology in degree 1 of Hilbert modular surfaces, using a result of Repka on tensor products of holomorphic and antiholomorphic discrete series representations [40]. A more interesting case, in which $G = \mathrm{GSp}(2, \mathbb{Q})$, is the subject of a forthcoming paper of the author with S. Kudla.

The present article is organized as follows. The first two sections recall the theory of Shimura varieties, toroidal compactifications, and canonical extensions, as generalized in [27], and prove the results in A above. In §3 we amplify the results of [28] and use them to relate automorphic forms to coherent cohomology classes, as in B. The notion of "representation with $\bar{\partial}$-cohomology" is introduced in §4, following suggestions of D. Vogan, and the $\bar{\partial}$-cohomology of discrete series and limits of discrete series is described.

The main theorems (C and D above) are stated in §5, together with some of their more notable consequences. The proof of Theorem (5.3) is completed in §6; it makes essential use of Langlands' theory of Eisenstein series. Finally, §7 contains the rationality criterion described above, and §8 works it out in some concrete examples.

This article represents a complete reworking of a previous manuscript, entitled "Automorphic forms of discrete type as coherent cohomology classes". The revised title reflects a change in emphasis, entirely inspired by the work of Blasius, Clozel, and Ramakrishnan on the arithmetic of Maass forms of Galois type [4]. The main result in [4] depends on the association to such a Maass form of a family of cuspidal automorphic representations of GSp(2) whose archimedean factors are nonholomorphic limits of discrete series. In the light of the previous manuscript, it was natural to ask whether such representations, which cannot be treated by the methods of $(\mathfrak{g}, K)$-cohomology, nevertheless define coherent cohomology classes. In discussions with Blasius and Ramakrishnan, we verified that this was indeed the case (Theorem (4.6.2)); a significant strengthening of the main result of [4] is an immediate consequence. Many of the ideas in the proof of Theorem (5.3) also first arose in these discussions. This revised version of the paper was largely written with a view to providing the foundations for our forthcoming joint paper [5].

The initial suggestion to look into the arithmeticity of automorphic forms of discrete series type was made by Takeyuki Oda, whose article [38] was an important influence on this work. Most of the ideas in the first version of this paper were developed while the author was a guest at the Institute for Advanced Study in the fall of 1983, and at the Ecole Normale Supérieure de Jeunes Filles in Montrouge, during the spring of 1985. The author thanks these institutions for their hospitality and the Sloan Foundation for its support during a visit to the latter institution. The ideas for the revised version of the paper were developed during a brief visit to Mathematical Sciences Research Institute in the spring of 1987, and during the conference on Representation Theory of Lie Groups and Automorphic Forms at Oberwolfach during the summer of that year. The writing of §§4 and 7 benefited from crucial advice of D. Vogan and R. Howe, respectively. Suggestions of P. Deligne resulted in the removal of some unnecessary assumptions from the results of §2. Of course, the results of §3 would have been impossible without the help of D. H. Phong. S. Kudla has on numerous occasions provided valuable suggestions, and specifically directed our attention to the article [54] of Wallach, which is crucial in many of the applications of Theorem (5.3). At various points in the writing, this paper has benefited from discussions between the author and A. Borel, L. Clozel, P. Garrett, J. S. Milne, S. Rallis, J. Schwermer, N. Wallach, and F. Williams. These mathematicians are gratefully acknowledged as is G. Harder for his consistent and generous encouragement.

Notation and conventions. The symbols $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_l$, and $\mathbb{Z}_l$ have their usual meanings. By $\mathbf{A}$ (resp. $\mathbf{A}^f$) we mean the ring of rational adeles (resp. of rational finite adeles). The group schemes $\mathrm{GL}(n)$ and $\mathbb{G}_m$ are denoted as usual. By $\bar{\mathbb{Q}}$ we always mean the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$.

If V and T are schemes over the scheme S, then $V(T)$ denotes the set of T-valued points of V; $V_T = V \times_S T$. If T is Spec(A) for some ring A, we often write $V(A)$ and V_A in place of $V(T)$ and V_T. If $S = \operatorname{Spec} k'$, where k' is a finite field extension of the field k, then $R_{k'/k}V$ is the scheme over k obtained by Weil's restriction of scalars functor. The structure sheaf of V is denoted $\mathcal{O}_V$.

If G is an algebraic group, then G^{ad}, G^{der}, G^{ab},x and Z_G are the adjoint group, the derived subgroup, the abelianization G/G^{der}, and the center of G, respectively. The Lie algebra of G is denoted $\mathfrak{g}$ or Lie(G); the enveloping algebra of $\mathfrak{g}$ is $U(\mathfrak{g})$, and the center of $U(\mathfrak{g})$ is written $Z(\mathfrak{g})$. The unipotent radical of G is denoted $R_u(G)$. If G is a topological group, then G^0 is its connected component containing the identity.

If X is a C^∞-manifold and V is a complex vector space, then $C^\infty(X, V)$ is the space of C^∞ functions on X with values in V. If X is an adelic group, then $C^\infty(X, V)$ is the space of V-valued functions on X, which are C^∞ (resp. locally constant) in the archimedean (resp. nonarchimedean) variables.

If X is a smooth algebraic (resp. complex analytic) variety, then Ω_X^k is the bundle (or sheaf) of algebraic (resp. holomorphic) differential k-forms on X. If X is a complex manifold, then $\Omega_X^{p,q}$ is the sheaf of C^∞ differential forms of Hodge type (p, q) on X.

If $\mathcal{E}$ is a vector bundle over the (algebraic or analytic) variety X, then $\Gamma(X, \mathcal{E})$ is the space of global sections of $\mathcal{E}$ over X. The same notation is used for C^∞ vector bundles. We make no notational distinction between $\mathcal{E}$ and its associated locally free sheaf; in particular, if X is an algebraic variety, then $H^*(X, \mathcal{E})$ denotes cohomology of the sheaf of sections of $\mathcal{E}$ in the Zariski topology.

Let $\underline{S}$ be the torus $R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_m)$; let $z, \bar{z} \in X_{\mathbb{C}}(\underline{S})$ be the characters which induce their namesakes on $\mathbb{C}^\times = \underline{S}(\mathbb{R}) \subset \underline{S}(\mathbb{C})$. A *Hodge structure* on a $\mathbb{Q}$-vector space V is a homomorphism $h: \underline{S} \to \mathrm{GL}(V_{\mathbb{R}})$. We write

$$V_{\mathbb{C}} = \bigoplus_{p,q} V^{p,q}, \qquad F^p V = \bigoplus_{p' \geq p} V^{p,q},$$

where $\underline{S}_{\mathbb{C}}$ acts on $V^{p,q}$ through the character $z^{-p}\bar{z}^{-q}$. Then $\overline{V^{p,q}} = V^{q,p}$.

If G is an algebraic group and $\rho: G \to \mathrm{GL}(V)$ is an algebraic representation, we often denote the representation (ρ, V), and use ρ and V interchangeably. If G is a topological group and V is a topological vector space, we use the same convention. If G is a reductive Lie group, $K_\infty \subset G$ is an algebraic subgroup containing a maximal compact subgroup, and (π, V) is a unitary representation of G, we often denote by π_0 or V_0 the $(\mathfrak{g}, K_\infty)$ module associated to π. Here $(\mathfrak{g}, K_\infty)$ modules are defined as in [12], with the following modification: since K_∞ typically contains the center of G and is thus not compact, we require that the K_∞-types occurring in the restriction of π to K_∞ be finite-dimensional *algebraic* representations of K_∞.

1. Shimura varieties and toroidal compactifications

(**1.1**) Let $\underline{S}$ be the real algebraic torus $R_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m$. Let (G, X) be a pair consisting of a connected reductive algebraic group $G/\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class of homomorphisms $h: \underline{S} \to G_{\mathbb{R}}$ satisfying the following conditions ([17]; cf. [25]):

(**1.1.1**) The Hodge structure on the Lie algebra $\mathfrak{g}$ of G, given by $\mathrm{ad} \circ h$, is of type $(0, 0) + (-1, 1) + (1, -1)$.

(**1.1.2**) The automorphism $\mathrm{ad}(h(i))$ of $G(\mathbb{R})$ induces a Cartan involution on $G^{\mathrm{der}}(\mathbb{R})^0$.

(**1.1.3**) Let $w: \mathbb{G}_{m,\mathbb{R}} \xrightarrow{\sim} \underline{S}$ be the canonical conorm map. The *weight map* $h \circ w: \mathbb{G}_{m,\mathbb{R}} \xrightarrow{\sim} G_{\mathbb{R}}$, whose image is (by (1.1.1)) central in $G_{\mathbb{R}}$, is defined over $\mathbb{Q}$.

(**1.1.4**) Let $Z'_G \subset Z_G$ be the maximal $\mathbb{Q}$-split torus of Z_G. Then $Z_G(\mathbb{R})/Z'_G(\mathbb{R})$ is compact.

We call such a (G, X) a *basic pair*. The space X has a natural $G(\mathbb{R})$-invariant complex structure. We define the associated Shimura variety as follows: If $K \subset G(\mathbf{A}^f)$ is an open compact subgroup, then

$$_K M(G, X)_{\mathbb{C}} \underset{\mathrm{def}}{=} G(\mathbb{Q})\backslash X \times G(\mathbf{A}^f)/K$$

is a (nonconnected) quasiprojective complex algebraic variety [3] and

$$M(G, X)_{\mathbb{C}} = \varprojlim_{K} {}_K M(G, X)_{\mathbb{C}}$$

is a pro-algebraic complex variety with continuous $G(\mathbf{A}^f)$-action. Thus $M(G, X)_{\mathbb{C}}$ is the set of complex points of the Shimura variety associated to (G, X). An open compact subgroup $K \subset G(\mathbf{A}^f)$ will be called a *level subgroup*.

(**1.1.5**) We refer to [17] or [25, §1] for the definition of the *reflex field* $E(G, X)$ of the basic pair (G, X). It is a theorem due in many cases to

Shimura and Deligne, and proved in general by Borovoi and Milne, that the variety $M(G, X)_{\mathbb{C}}$ has a *canonical model* $M(G, X)$ over the reflex field $E(G, X)$ [47], [17], [13], [34]. The definition of the canonical model can be found in [17]; here we note only that the action of $G(\mathbf{A}^f)$ on $M(G, X)_{\mathbb{C}}$ descends to $M(G, X)$ over $E(G, X)$.

More generally, the Langlands conjecture, proved by Milne and Borovoi [34], [13], states that $M(G, X)^{\tau}_{\mathbb{C}}$ is a Shimura variety $M(G^{(\tau)}, X^{(\tau)})_{\mathbb{C}}$ for every $\tau \in \mathrm{Aut}(\mathbb{C})$, and determines the basic pair $(G^{(\tau)}, X^{(\tau)})$.

(**1.2**) Let D be a connected component of X and let $G(\mathbb{R})^+ \subset G(\mathbb{R})$ be the subgroup which stabilizes D. For any subgroup $S \subset G(\mathbb{R})$ let $S^+ = S \cap G(\mathbb{R})^+$. Let $G_0 = G^{\mathrm{der}}(\mathbb{R})^0$. The action of G_0 on D identifies D with the Riemannian symmetric space associated to G_0. Let $\Gamma \subset G(\mathbb{Q})^+$ be an arithmetic subgroup. We assume Γ is neat [7]: If $\rho\colon G \to \mathrm{GL}(V)$ is a faithful representation, and $\gamma \in \Gamma$, then the group generated by the eigenvalues of $\rho(\gamma)$ contains no root of unity other than 1. Let $M = M_\Gamma$ be the quotient $\Gamma\backslash D$. Then M_Γ is a smooth quasiprojective complex variety.

We know that ${}_K M(G, X)_{\mathbb{C}} = G(\mathbb{Q})^+\backslash D \times G(\mathbf{A}^f)/K$ for any K [17, 2.1]. Let $\{\gamma\}$ be a set of representatives of the double cosets $G(\mathbb{Q})^+\backslash G(\mathbf{A}^f)/K$; then

$$(1.2.1) \qquad {}_K M(G, X)_{\mathbb{C}} = \coprod_{\{\gamma\}} M_{\Gamma(\gamma)}, \qquad \Gamma(\gamma) = G(\mathbb{Q})^+ \cap \gamma\Gamma\gamma^{-1}.$$

We define a *neat* K as in [27, 1.1]. For our purposes, it suffices to mention that every compact open subgroup $K \subset G(\mathbf{A}^f)$ evidently contains a neat subgroup of finite index, and that, if K is neat in (1.2.1), then so is $\Gamma(\gamma)$ for all γ.

(**1.2.2**) We now describe the toroidal compactifications of ${}_K M(G, X)_{\mathbb{C}}$, in the framework of [2]. We begin by recalling the construction in [2] of the toroidal compactifications of the connected components of ${}_K M(G, X)_{\mathbb{C}}$. Fix D and a neat arithmetic subgroup $\Gamma \subset G(\mathbb{Q})^+$ as above. To the pair (D, Γ) is associated, as in [3], a collection $\{F\}$ of *rational boundary components*. Let $P_F \subset G$ be the maximal $\mathbb{Q}$-parabolic such that $P_F(\mathbb{R})^+$ stabilizes F. Choose a Levi decomposition $P_F = M_F \cdot W_F$, where W_F is the unipotent radical of P_F; let U_F be the center of W_F. The logarithm map identifies U_F with the $\mathbb{Q}$-vector space $\mathrm{Lie}(U_F(\mathbb{Q}))$.

There is naturally a $\mathbb{Q}$-rational positive-definite quadratic form $\langle\ ,\ \rangle$ on U_F. Inside $U_F(\mathbb{R})$ is a distinguished open convex cone C_F, selfadjoint with respect to $\langle\ ,\ \rangle$. (For details and definitions, cf. [2, III, §4].) Let $U_F(\mathbb{Z}) = U_F(\mathbb{Q}) \cap \Gamma$; $U_F(\mathbb{Z})$ is a lattice in $U_F(\mathbb{R})$.

Let $P' \subset P_F$ denote the centralizer of U_F in P_F. Let $D_F = U_F(\mathbb{C}) \cdot \beta(D) \subset \check{M}(\mathbb{C})$, where β is the Borel imbedding defined above. Then the action of $\Gamma'_F \underset{\text{def}}{=} \Gamma \cap P'(\mathbb{Q})$ on D extends to D_F. Let $M'_F = \Gamma'_F \backslash D_F$, and let T_F be the $\mathbb{Q}$-split torus with character group $X(T_F) = \mathrm{Hom}(U_F(\mathbb{Z}), \mathbb{Z})$; thus $T_F(\mathbb{C}) \cong U_F(\mathbb{C})/U_F(\mathbb{Z})$ acts holomorphically on M'_F. It was proved by Brylinski ([14]; cf. [27, 1.2]) that M'_F has the structure of an algebraic variety, and that the action of $T_F(\mathbb{C})$ on M'_F defines a $T_F(\mathbb{C})$-fibration $\pi_2: M'_F \to A_F$ where A_F is a smooth quasiprojective algebraic variety.

Let $\sigma \subset U_F(\mathbb{R})$ be a rational polyhedral cone (rpc); i.e., a subset of the form $\{\sum_{i=1}^{a} |\lambda_i v_i| \lambda_i \geq 0\}$, where $v_i \in U_F(\mathbb{Q})$, $i = 1, \cdots, a$. The dual space $X(T_F) \otimes \mathbb{R} \cong \mathrm{Hom}(U_F(\mathbb{R}), \mathbb{R})$ contains the dual cone

$$\check{\sigma} = \{\lambda \in X(T_F) \otimes \mathbb{R} | \lambda(v) \geq 0 \ \forall v \in \sigma\}.$$

As in [30, §1], let $T_\sigma = \mathrm{Spec}\, \mathbb{Q}[X(T_F) \cap \check{\sigma}]$; then T_F imbeds naturally in T_σ, and the action of T on itself extends to an action on T_σ. In this way we define a 1-1 correspondence between rpc's $\sigma \subset U_F(\mathbb{R})$ and normal equivariant affine imbeddings $T_F \hookrightarrow T_\sigma$. More generally, if $\sigma = \bigcup \sigma'$ is a finite simplicial decomposition, then the $T_{\sigma'}$ patch together to an equivariant torus imbedding $T_{\{\sigma'\}}$, and the natural maps $T_{\sigma'} \to T_\sigma$ patch together to a proper surjective T-equivariant morphism $T_{\{\sigma'\}} \to T_\sigma$.

Given a rpc $\sigma \subset U_F(\mathbb{R})$, we define $(M'_F)_\sigma = M'_F \times^{T_F} T_\sigma$, and let $D_{F,\sigma}$ denote the interior of the closure in $(M'_F)_\sigma$ of $\Gamma'_F \backslash D$. We let $\pi_{2,\sigma}: D_{F,\sigma} \to A_F$ be the natural projection. If $\sigma = \bigcup \sigma'$ as above, we define $(M'_F)_{\{\sigma'\}}$ and $D_{F,\{\sigma'\}}$ analogously, then there are proper morphisms

$$(1.2.2.1) \qquad D_{F,\{\sigma'\}} \to D_{F,\sigma} \quad (\text{resp. } (M'_F)_{\{\sigma'\}} \to (M'_F)_\sigma)$$

of analytic spaces (resp. algebraic varieties) over A_F.

(**1.2.3**) A toroidal compactification $M_\Gamma \hookrightarrow M_{\Gamma,\Sigma}$ is associated to a collection $\Sigma = \bigcup_F \Sigma_F$, where each Σ_F is a collection of rpc's $\{\sigma\}$, $\sigma \subset U_F(\mathbb{R})$, satisfying a long list of hypotheses (cf. [2, p. 252]). We refer to Σ as a Γ-*admissible collection of polyhedra*. The Σ's are partially ordered by the relation of *refinement*; Σ' is a refinement of Σ if every $\sigma' \in \Sigma'$ is contained in exactly one $\sigma \in \Sigma$, and $\forall \sigma \in \Sigma$ the set $\{\sigma' \in \Sigma' | \sigma' \subset \sigma\}$ is a finite simplicial decomposition of σ. The spaces $M_{\Gamma,\Sigma}$ are in general only algebraic spaces over $\mathbb{C}$.

For our purposes, it suffices to note the following:

(**1.2.3.1**) For each F and each $\sigma \in \Sigma_F$, the natural map $\Gamma'_F \backslash D \to M_\Gamma$ extends to a local analytic isomorphism $\varphi_{F,\sigma}: D_{F,\sigma} \to M_{\Gamma,\Sigma}$.

(**1.2.3.2**) The union of the images of the $\varphi_{F,\sigma}$ form an open covering of $M_{\Gamma,\Sigma}$.

(**1.2.3.3**) Any Σ has a refinement Σ' such that $M_{\Gamma,\Sigma'}$ is smooth and projective, and such that $M_{\Gamma,\Sigma'} - M_{\Gamma}$ is a divisor with normal crossings. Such a compactification will be called SNC.

If Σ' is a refinement of Σ, then there is a natural proper surjective morphism $\pi_{\Sigma',\Sigma}: M_{\Gamma,\Sigma'} \to M_{\Gamma,\Sigma}$ consistent with (1.2.2.1) and (1.2.3.1).

(**1.2.4**) In [27, 2.5] we construct an adelic version of the above theory. Fix a level subgroup $K \subset G(\mathbf{A}^f)$, and write ${}_KM = {}_KM(G, X)_{\mathbb{C}}$. For each standard $\mathbb{Q}$-rational maximal parabolic subgroup P, relative to a fixed minimal $\mathbb{Q}$-parabolic P_0, we take a collection Σ_P of rpc's in $R_uP(\mathbb{R})$, satisfying a long list of axioms [27, 2.5.1]. Let $\Sigma = \bigcup_P \Sigma_P$; then there is a toroidal compactification ${}_KM \hookrightarrow {}_KM_{\Sigma}$; such toroidal compactifications are called *admissible* in [27] and we call Σ a *K-admissible collection of polyhedra.*

Let F_P be the rational boundary component of D fixed by $P(\mathbb{R})^+$. Each $\sigma \in \Sigma_P$ defines an analytic variety $D_{P,\sigma}$ isomorphic to a $D_{F,\sigma}$ as above, and an analytic morphism $\varphi_{P,\sigma}: D_{P,\sigma} \to {}_KM_{\Sigma}$, satisfying the analogues of properties (1.2.3.1)–(1.2.3.3). Corresponding to the decomposition (1.2.1) we have a decomposition

$$(1.2.4.1) \qquad {}_KM_{\Sigma} = \coprod_{\{\gamma\}} M_{\Gamma(\gamma),\Sigma(\gamma)},$$

where $M_{\Gamma(\gamma),\Sigma(\gamma)}$ is a toroidal compactification of $M_{\Gamma(\gamma)}$.

(1.2.5) Remark. Suppose the defining data Σ are *projective* in the sense of Tai [2, IV, §2], and *equivariant* in the sense of [27, 2.7]. Then [27, Proposition 2.8], the toroidal compactification ${}_KM_{\Sigma}$ is defined over $E(G, X)$, as is the divisor ${}_KZ_{\Sigma} = {}_KM_{\Sigma} - {}_KM$. As remarked in [27], there exist projective and equivariant data Σ which define SNC compactifications.

(**1.3**) We now describe some simple functorial properties of the ${}_KM_{\Sigma}$.

(**1.3.1**) Suppose Σ is a K-admissible collection of polyhedra and K' is an open subgroup of K. Then Σ is K'-admissible, and there is a map [27, (2.5.7) (c)]

$$t_{K',K}: {}_{K'}M(G, X)_{\Sigma} \to {}_KM(G, X)_{\Sigma}.$$

(**1.3.2**) Let $h \in G(\mathbf{A}^f)$; then $K^h = h^{-1}Kh$ is also neat. We can define a K^h-admissible collection of polyhedra Σ^h such that the natural isomorphism ${}_{K^h}M(G, X) \xrightarrow{\sim} {}_KM(G, X)$ given by right multiplication by h extends to an isomorphism of algebraic spaces $t_h: {}_{K^h}M_{\Sigma^h} \xrightarrow{\sim} {}_KM_{\Sigma}$.

(1.3.3) Let $(G^{\#}, X^{\#}) \subset (G, X)$ be another basic pair, and define $K^{\#}$ and $\Sigma^{\#}$ as in [27, §3]. Proposition 3.4 of [27] provides a morphism $\psi_\Sigma : M^{\#}_{\Sigma^{\#}} \to M_\Sigma$, extending the natural map $\psi : M^{\#} \to M$. If Σ is projective and equivariant, then so is $\Sigma^{\#}$, and the morphism ψ_Σ is rational over $E(G^{\#}, X^{\#})$.

(1.3.4) In general, if Σ is projective and equivariant, then the morphisms ψ_Σ, $t_{K',K}$ and t_h defined above are rational over $E(G, X)$ (or $E(G^{\#}, X^{\#})$).

2. Cohomology of extensions of automorphic vector bundles

(2.0) In most of what follows we fix a point $h \in X$, and let $K_\infty \subset G(\mathbb{R})$ be the stabilizer of h. Much confusion will be avoided if the reader bears in mind that $K_\infty \supset Z_G(\mathbb{R})$, and is thus *noncompact* in general. A representation of K_∞ will always be assumed *algebraic*, unless otherwise indicated.

The point h defines by (1.1.1) a Hodge decomposition on $\mathfrak{g}$:

(2.0.1) $$\mathfrak{g}_\mathbb{C} = \mathfrak{k}_{\infty,\mathbb{C}} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^- \underset{\text{def}}{=} \mathfrak{g}_\mathbb{C}^{(0,0)} \oplus \mathfrak{g}_\mathbb{C}^{(-1,1)} \oplus \mathfrak{g}_\mathbb{C}^{(1,-1)}.$$

Here $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) corresponds to the holomorphic (resp. antiholomorphic) tangent space of X at h. Let $\mathscr{P}_h$ be the parabolic subgroup of G with Lie algebra $F^0\mathfrak{g}$; its unipotent radical $R_u\mathscr{P}_h$ has Lie algebra $\mathfrak{p}^-$.

Choose a maximal torus $H \subset K_\infty$, and let $\mathfrak{h}$ be its Lie algebra. Then $\mathfrak{h}_\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$ as well as of $\mathfrak{k}_{\infty,\mathbb{C}}$. Let R be the set of roots of $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$, and let R_c (resp. R_n) denote the compact (resp. noncompact) roots. Once and for all we choose a set R^+ of positive roots in R, such that $R_n^+ \underset{\text{def}}{=} R_n \cap R^+$ corresponds to the root spaces in $\mathfrak{p}^+$; we write $R_c^+ \underset{\text{def}}{=} R_c \cap R^+$. Let

$$\rho = \tfrac{1}{2} \sum_{\alpha \in R^+} \alpha, \quad \rho_n = \tfrac{1}{2} \sum_{\alpha \in R_n^+} \alpha, \quad \rho_c = \rho - \rho_n.$$

(2.1) Automorphic vector bundles. For what follows, cf. [25], especially §3. Let $\check{M}(\mathbb{C})$ be the compact dual symmetric space of D. We may define $\check{M}(\mathbb{C})$ as $G(\mathbb{C})/\mathscr{P}_h(\mathbb{C})$. Let $\sigma : \mathscr{P}_h(\mathbb{C}) \to \mathrm{GL}(V_\sigma)$ be a finite-dimensional algebraic representation over $\mathbb{C}$. This defines, by the usual procedure, a $G(\mathbb{C})$-homogeneous vector bundle $\check{E} = \check{E}_\sigma$ on $\check{M}(\mathbb{C})$. Let $\beta : X \hookrightarrow \check{M}(\mathbb{C})$ be the Borel imbedding, defined as in [25, 3.1]; it is the unique $G(\mathbb{R})$-equivariant map whose restriction to D is the open immersion defined above. For any level subgroup $K \subset G(\mathbf{A}^f)$,

$$[\check{E}_\sigma] = G(\mathbb{Q})\backslash\beta^*(\check{E}_\sigma) \times G(\mathbf{A}^f)/K$$

is an algebraic vector bundle over ${}_KM(G, X)_\mathbb{C}$ [3, §10], and $[\check{E}_\sigma] = \varprojlim_K [\check{E}_\sigma]_K$ is a $G(\mathbf{A}^f)$-homogeneous algebraic vector bundle over $M(G, X)_\mathbb{C}$.

The compact dual symmetric space $\check{M}(\mathbb{C})$ has a natural rational structure over the reflex field $E(G, X)$, described in [25, §3]. One of the main theorems of [25] is the following:

(2.1.1) Theorem. *The functor* $\mathscr{V} \mapsto [\mathscr{V}]$, *from G-homogeneous vector bundles on* $\check{M}$ *to* $G(\mathbf{A}^f)$*-homogeneous vector bundles over* $M(G, X)$, *is rational over* $E(G, X)$.

Milne has proved a strengthening of Theorem (2.1.1) [35] in the context of the Langlands conjecture (cf. (1.1.5)).

A bundle of the form $[\mathscr{V}]$, with $\mathscr{V}$ as above, is called an *automorphic vector bundle.* The automorphic vector bundle $[\mathscr{V}]$ is called *fully decomposed* if it is of the form $[E_\sigma]$, where σ factors through the reductive quotient $K_\infty(\mathbb{C})$ of $\mathscr{P}_h(\mathbb{C})$, and *irreducible* if σ is an irreducible representation.

(**2.2**) Fix a neat level subgroup $K \subset G(\mathbf{A}^f)$, and let ${}_KM = {}_KM(G, X)$. Let $\Sigma = \bigcup \Sigma_P$ be as in (1.2), and let $j_\Sigma \colon {}_KM \hookrightarrow {}_KM_\Sigma$ be the corresponding toroidal compactification; for each γ, let $\Sigma(\gamma) = \bigcup \Sigma(\gamma)_F$ be the combinatorial data defining a toroidal compactification $j_{\Sigma(\gamma)} \colon M_{\Gamma(\gamma)} \hookrightarrow M_{\Gamma(\gamma), \Sigma(\gamma)}$ (cf. (1.2.1)). We are going to define two functors from the category of automorphic vector bundles on ${}_KM$ to the category of vector bundles on ${}_KM_\Sigma$.

Let $\mathscr{V}$ be a G-homogeneous vector bundle over $\check{M}$, and let $\mathscr{V}_F^0$ be the restriction of $\mathscr{V}$ to D_F; let $\mathscr{V}_F$ be the vector bundle $\Gamma'_F\backslash\mathscr{V}_F^0$ over $M'_F = \Gamma'_F\backslash D_F$. As explained in 4.1 of [27], $\mathscr{V}_F \cong \pi_2^*(\mathscr{V}_F^A)$ for some vector bundle $\mathscr{V}_F^A$ over A_F; we let $\mathscr{V}_{F,\sigma} = \pi_{2,\sigma}^*(\mathscr{V}_F^A)$ if $\sigma \in \Sigma(\gamma)_F$ (notation (1.2.2)). The *canonical extension* of $[\mathscr{V}]$ is the unique subsheaf $[\mathscr{V}]^{\mathrm{can}}$ of $j_{\Sigma,*}([\mathscr{V}])$ over ${}_KM_\Sigma$ such that, for each component $M_{\Gamma(\gamma),\Sigma(\gamma)}$ and each $\sigma \in \Sigma(\gamma)_F$, there exist isomorphisms

$$f_\sigma \colon \varphi_{F,\sigma}^*[\mathscr{V}]^{\mathrm{can}} \xrightarrow{\sim} \mathscr{V}_{F,\sigma} \tag{2.2.1}$$

satisfying the obvious compatibility relations.

(2.2.2) Proposition [27, *Theorem* 4.2]. *Let* ${}_KM_\Sigma$ *be an admissible toroidal compactification of* $M_\mathbb{C}$. *Then the following hold:*

(i) *Any automorphic vector bundle* $[\mathscr{V}]$ *over* ${}_KM_\mathbb{C}$ *has a canonical extension* $[\mathscr{V}]^{\mathrm{can}}$ *over* ${}_KM_\Sigma$.

(ii) *The functor* $[\mathscr{V}] \mapsto [\mathscr{V}]^{\mathrm{can}}$ *is exact and commutes with tensor products and* Hom. *Moreover,* $[\mathscr{O}_{\check{M}}]_\Sigma \cong \mathscr{O}_{{}_KM_\Sigma}$ [27, *Proposition* 4.4].

(iii) *Suppose* ${}_KM_\Sigma$ *is defined by a projective and equivariant* Σ, *in the sense of* [27, 2.7] *and* [2, IV]. *Then the functor* $[\mathscr{V}] \mapsto [\mathscr{V}]^{\mathrm{can}}$ *preserves fields of definition. In other words, the functor* $\mathscr{V} \mapsto [\mathscr{V}]^{\mathrm{can}}$, *taking G-homogeneous vector bundles over* $\check{M}$ *to vector bundles over* ${}_KM_\Sigma$, *is rational over* $E(G, X)$.

(**2.2.3**) Let $\mathscr{I}(Z_\Sigma) \subset \mathscr{O}_{{}_KM_\Sigma}$ be the ideal sheaf defining the divisor $Z_\Sigma = {}_KM_\Sigma - {}_KM$. Since ${}_KM_\Sigma$ is normal, $\mathscr{I}(Z_\Sigma)$ is an invertible sheaf. Given any automorphism vector bundle $[\mathscr{V}]$, the *subcanonical extension* of $[\mathscr{V}]$ is the vector bundle $[\mathscr{V}]^{\mathrm{sub}} = [\mathscr{V}]^{\mathrm{can}} \otimes \mathscr{I}(Z_\Sigma)$. Let $[\mathscr{V}]_Z$ denote the restriction to Z_Σ of $[\mathscr{V}]^{\mathrm{can}}$. Then we have an exact sequence of sheaves on M_Σ

$$0 \to [\mathscr{V}]^{\mathrm{sub}} \to [\mathscr{V}]^{\mathrm{can}} \to [\mathscr{V}]_Z \to 0. \tag{2.2.4}$$

If Σ is projective and equivariant and $\mathscr{V}$ is defined over the field L as a homogeneous vector bundle over $\check{M}$, then it follows from Proposition (2.2.2) and Remark (1.2.5) that (2.2.4) is rational over L.

When necessary, we write $[\mathscr{V}]^{\mathrm{sub}}_\Sigma$, $[\mathscr{V}]^{\mathrm{can}}_\Sigma$, $[\mathscr{V}]_{Z_\Sigma}$. If Σ' is a refinement of Σ, let $\pi_{\Sigma',\Sigma}\colon {}_KM_{\Sigma'} \to {}_KM_\Sigma$ be the natural map. Then $\pi^*_{\Sigma',\Sigma}\mathscr{I}(Z_\Sigma)$ is naturally isomorphic to $\mathscr{I}(Z_{\Sigma'})$. From this remark and (2.2.1) it follows that we have natural isomorphisms

$$\pi^*_{\Sigma',\Sigma}[\mathscr{V}]^{\mathrm{can}}_\Sigma \cong [\mathscr{V}]^{\mathrm{can}}_{\Sigma'}, \qquad \pi^*_{\Sigma',\Sigma}[\mathscr{V}]^{\mathrm{sub}}_\Sigma \cong [\mathscr{V}]^{\mathrm{sub}}_{\Sigma'}. \tag{2.2.5}$$

The important property of $[\mathscr{V}]^{\mathrm{sub}}$ is the following:

(**2.2.6**) **Proposition** (*Mumford* [36]). (i) *If* ${}_KM_\Sigma$ *is SNC, then for any integer* $r > 0$, $[\Omega^r_{\check{M}}]^{\mathrm{can}} \cong \Omega^r_{{}_KM_\Sigma}(\log Z)$, *where* Z *is the divisor* ${}_KM_\Sigma - {}_KM$, *and* $\Omega^{\cdot}_{{}_KM_\Sigma}(\log Z)$ *is the logarithmic De Rham complex of Deligne* [16]. *In particular, if* $n = \dim X$, *then* $[\Omega^n_{\check{M}}]^{\mathrm{sub}} \cong \Omega^n_{{}_KM_\Sigma}$.

(ii) (*Kempf* [30, §3]). *For any* Σ, ${}_KM_\Sigma$ *is Cohen-Macauley, and* $[\Omega^n_{\check{M}}]^{\mathrm{sub}}$ *is isomorphic to the dualizing sheaf* $\mathbb{K}_{{}_KM_\Sigma}$.

(iii) *Let* Σ' *be a refinement of* Σ. *Then the diagram*

$$\begin{array}{ccc} \pi^*_{\Sigma',\Sigma}([\Omega^n_{\check{M}}]^{\mathrm{sub}}_\Sigma) & \xrightarrow{\sim} & \pi^*_{\Sigma',\Sigma}(\mathbb{K}_{{}_KM_\Sigma}) \\ \downarrow & & \downarrow \\ [\Omega^n_{\check{M}}]^{\mathrm{sub}}_{\Sigma'} & \xrightarrow{\sim} & \mathbb{K}_{{}_KM_{\Sigma'}} \end{array}$$

is commutative.

Proof. We need only justify (iii); but (iii) follows immediately from [30, §2, Theorem 9, III(d) and §3, Theorem 14(b)] and (2.2.1) above.

Let $\mathbb{K} = \mathbb{K}_{{}_K M} = \Omega^n_{{}_K M}$, and write $\mathbb{K}_\Sigma = \mathbb{K}_{{}_K M_\Sigma}$ for the dualizing sheaf of ${}_K M_\Sigma$, whether or not ${}_K M_\Sigma$ is smooth. Thus $\mathbb{K}^{\mathrm{sub}} \cong \mathbb{K}_\Sigma$.

(2.3) Corollary. *For any vector bundle $\mathscr{E}$ over ${}_K M$, let $\mathscr{E}' = \mathbb{K} \otimes \mathscr{E}^*$. For any automorphic vector bundle $[\mathscr{V}]$, and any $q \in \mathbb{Z}$, the cup product*

$$(2.3.1)\quad H^{n-q}({}_K M_\Sigma, [\mathscr{V}]^{',\mathrm{sub}}) \otimes H^q({}_K M_\Sigma, [\mathscr{V}]^{\mathrm{can}}) \to H^n({}_K M_\Sigma, \mathbb{K}^{\mathrm{sub}}) \cong \mathbb{C}$$

is a nondegenerate pairing (Serre duality). If Σ is projective and equivariant, then (2.3.1) *is rational over any base field k over which ${}_K M$ and $[\mathscr{V}]$ are defined.*

Proof. It follows from (2.2.4) that

$$\begin{aligned}\Omega^n_{{}_K M_\Sigma} \otimes ([\mathscr{V}]^{\mathrm{can}})^* &\cong \mathbb{K}^{\mathrm{can}} \otimes \mathscr{I}(Z_\Sigma) \otimes ([\mathscr{V}]^{\mathrm{can}})^* \\ &= \mathbb{K}^{\mathrm{can}} \otimes ([\mathscr{V}]^{\mathrm{can}})^* \otimes \mathscr{I}(Z_\Sigma) \\ &= (\mathbb{K} \otimes [\mathscr{V}^*])^{\mathrm{can}} \otimes \mathscr{I}(Z_\Sigma) = [\mathscr{V}]^{',\mathrm{sub}},\end{aligned}$$

where the next to last equality follows from Proposition (2.2.2)(ii). The corollary, including the final isomorphism in (2.3.1), now follows from Serre duality.

The following proposition is the starting point for the theory:

(2.4) Proposition. *Let K be a neat level subgroup of $G(\mathbf{A}^f)$, and let Σ be a K-admissible collection of polyhedra. Let $[\mathscr{V}]$ be an automorphic vector bundle over ${}_K M = {}_K M(G, X)$. Let Σ' be a K-adapted refinement of Σ* [27, §2]. *Then the natural homomorphisms of sheaf cohomology*

$$(2.4.1)\quad \begin{aligned} H^q({}_K M_\Sigma, [\mathscr{V}]^{\mathrm{can}}_\Sigma) &\to H^q({}_K M_{\Sigma'}, [\mathscr{V}]^{\mathrm{can}}_{\Sigma'}), \\ H^q({}_K M_\Sigma, [\mathscr{V}]^{\mathrm{sub}}_\Sigma) &\to H^q({}_K M_{\Sigma'}, [\mathscr{V}]^{\mathrm{sub}}_{\Sigma'}) \end{aligned}$$

are isomorphisms for $q = 0, \cdots, n$.

(2.4.2) Remark. The case $q = 0$ was observed by Mumford [36, Proposition 3.3].

Proof. It suffices to verify (2.4.1) on each connected component of ${}_K M$. Thus, let $\Gamma \subset G(\mathbb{Q})^+$ be a neat arithmetic subgroup, and define M_Γ as in (1.2). Let Σ be a Γ-admissible collection of polyhedra, Σ' a refinement of Σ, and $\pi: M_{\Gamma,\Sigma'} \to M_{\Gamma,\Sigma}$ the natural map. The canonical (resp. subcanonical) extension of (the restriction to M_Γ of) $[\mathscr{V}]$ to $M_{\Gamma,\Sigma}$ is denoted $[\mathscr{V}]^{\mathrm{can}}_\Sigma$ (resp. $[\mathscr{V}]^{\mathrm{sub}}_\Sigma$); likewise for $M_{\Gamma,\Sigma'}$. Using the Leray spectral sequence for the map $\pi_{\Sigma',\Sigma}$, we see it suffices to prove the following statements:

$$\pi_*[\mathscr{V}]^{\text{can}}_{\Sigma'} \cong [\mathscr{V}]^{\text{can}}_{\Sigma}, \tag{2.4.3}$$

$$\pi_*[\mathscr{V}]^{\text{sub}}_{\Sigma'} \cong [\mathscr{V}]^{\text{sub}}_{\Sigma}, \tag{2.4.4}$$

$$R^i\pi_*[\mathscr{V}]^{\text{can}}_{\Sigma'} = 0, \qquad i > 0, \tag{2.4.5}$$

$$R^i\pi_*[\mathscr{V}]^{\text{sub}}_{\Sigma'} = 0, \qquad i > 0. \tag{2.4.6}$$

Now (2.4.3) is (a) of [27, Lemma 4.2.4]. We prove (2.4.5). By GAGA [44] which, as remarked in [27, 4.1.2], is valid for proper maps between algebraic spaces, we may calculate $R^i\pi_*[\mathscr{V}]^{\text{can}}_{\Sigma'}$ in the analytic category. The question is local on $M_{\Gamma,\Sigma}$. Thus, let $p \in M_{\Gamma,\Sigma}$, and suppose p is in the image of $\pi_{F,\sigma}\colon D_{F,\sigma} \to M_{\Gamma,\Sigma}$ for some $\sigma \in \Sigma$ (1.2.3.1). Let $\bar{B}$ be an open subset of $D_{F,\sigma}$ with the property that $\pi_{F,\sigma}$ is an analytic isomorphism between $\bar{B}$ and a neighborhood B of p in $M_{\Gamma,\Sigma}$. It suffices to calculate $R^i\pi_*[\mathscr{V}]^{\text{can}}_{\Sigma'}$ over B.

The refinement Σ' of Σ represents σ as the finite union $\bigcup \sigma'$. Define $(M'_F)_\sigma$, $(M'_F)_{\{\sigma'\}}$, and $D_{F,\{\sigma'\}}$ as in (1.2.2), and let $h'\colon D_{F,\{\sigma'\}} \to D_{F,\sigma}$ and $h\colon (M'_F)_{\{\sigma'\}} \to (M'_F)_\sigma$ be the morphisms (1.2.2.1). We denote by $\pi_{\{\sigma'\}}$ (resp. π_σ) the natural maps $(M'_F)_{\{\sigma'\}} \to A_F$ (resp. $(M'_F)_\sigma \to A_F$). By (2.2.1), we have an isomorphism $\varphi^*_{F,\sigma}[\mathscr{V}]^{\text{can}}_{\Sigma} \xrightarrow{\sim} \mathscr{V}_{F,\sigma} \underset{\text{def}}{=} \pi^*_{2,\sigma}(\mathscr{V}^A_F)$ for some vector bundle $\mathscr{V}^A_F$ over A_F. Let $\mathscr{W}_{\{\sigma'\}} = \pi^*_{\{\sigma'\}}(\mathscr{V}^A_F)$ and $\mathscr{W}_\sigma = \pi^*_\sigma(\mathscr{V}^A_F)$.

There is a commutative diagram

$$\begin{array}{ccccccc}
\pi^{-1}(B) & \xleftarrow[\sim]{\varphi_{F,\{\sigma'\}}} & h'^{-1}(\bar{B}) & \hookrightarrow & D_{F,\{\sigma'\}} & \longrightarrow & (M'_F)_{\{\sigma'\}} \\
\downarrow{\scriptstyle \pi} & & \downarrow & & \downarrow{\scriptstyle h'} & & \downarrow{\scriptstyle h} \\
B & \xleftarrow[\sim]{\varphi_{F,\sigma}} & \bar{B} & \hookrightarrow & D_{F,\sigma} & \longrightarrow & (M'_F)_\sigma
\end{array} \tag{2.4.7}$$

Since h is proper, we may apply GAGA again. It thus suffices to prove

$$R^q h_*(\mathscr{W}_{\{\sigma'\}}) = 0, \qquad q > 0, \tag{2.4.8}$$

where now h is a morphism of algebraic varieties over A_F.

Since (2.4.8) is local on $(M'_F)_\sigma$, we may replace A_F by a small affine open subset Y. Thus we may assume that $\mathscr{V}^A_F$ is the structure sheaf $\mathscr{O}_Y$ and that $\pi_2^{-1}(Y) \cong T_F \times Y$. If we denote by h_T the natural map

$T_{F,\{\sigma'\}} \to T_{F,\sigma}$, then (2.4.8) translates into the statement

$$R^q h_{T,*}(\mathscr{O}_{T_{F,\{\sigma'\}}}) = 0, \qquad q > 0. \tag{2.4.9}$$

But (2.4.9) is just (c) of Corollary (1), p. 44 of [30], where we take the functions g and f to be identically zero. This completes the proof of (2.4.5).

It remains to prove (2.4.4) and (2.4.6). Let $\mathscr{F} = \mathscr{V} \otimes (\Omega^n_{\check{M}})^*$, so that $[\mathscr{V}]^{\text{sub}}_{\Sigma'} \cong [\mathscr{F}]^{\text{can}}_{\Sigma'} \otimes \mathbb{K}_{\Sigma'}$ and $[\mathscr{V}]^{\text{sub}}_{\Sigma} \cong [\mathscr{F}]^{\text{can}}_{\Sigma} \otimes \mathbb{K}_{\Sigma}$. By (iii) of Proposition 2.2.6, it suffices to prove:

(2.4.10) The natural map $[\mathscr{F}]^{\text{can}}_{\Sigma} \otimes \mathbb{K}_{\Sigma} \to \pi_*(\pi^*([\mathscr{F}]^{\text{can}}_{\Sigma}) \otimes \mathbb{K}_{\Sigma'})$ is an isomorphism;

$$R^q \pi_*(\pi^*([\mathscr{F}]^{\text{can}}_{\Sigma}) \otimes \mathbb{K}_{\Sigma'}) = 0, \qquad q > 0. \tag{2.4.11}$$

As in the proof of (2.4.5), the problem is local in a neighborhood of the divisor Z_Σ. Proceeding as above, we may thus replace $\mathscr{F}$ by the structure sheaf $\mathscr{O}$, and the morphism ${}_KM_{\Sigma'} \to {}_KM_{\Sigma}$ by the map $T_{F,\{\sigma'\}} \to T_{F,\sigma}$. Statements (2.4.10) and (2.4.11) then reduce to a special case of (d) of [30, Corollary 1, p. 44].

(**2.5**) The functorial properties of the toroidal compactifications, described in (1.3), have their counterparts for canonical and subcanonical extensions. For any level subgroup $K \subset G(\mathbf{A}^f)$, we let ${}_K[\mathscr{V}]$ be $[\mathscr{V}]$ viewed as a vector bundle over ${}_KM(G, X)$. Fix one such K and a K-adapted Σ.

(**2.5.1**) Let $K' \subset K$ and $t_{K',K}: {}_{K'}M(G,X)_\Sigma \to {}_KM(G,X)_\Sigma$ be as in (1.3.1). Then

$$t^*_{K',K}({}_K[\mathscr{V}]^{\text{can}}_{\Sigma}) \cong {}_{K'}[\mathscr{V}]^{\text{can}}_{\Sigma'}, \qquad t^*_{K',K}({}_K[\mathscr{V}]^{\text{sub}}_{\Sigma}) \cong {}_{K'}[\mathscr{V}]^{\text{sub}}_{\Sigma'},$$

canonically.

(**2.5.2**) Let h, K^h, Σ^h and $t_h: {}_{K^h}M_{\Sigma^h} \xrightarrow{\sim} {}_KM_\Sigma$ be as in (1.3.2). Then

$$t^*_h({}_K[\mathscr{V}]^{\text{can}}_{\Sigma}) \cong {}_{K^h}[\mathscr{V}]^{\text{can}}_{\Sigma^h}, \qquad t^*_h({}_K[\mathscr{V}]^{\text{sub}}_{\Sigma}) \cong {}_{K^h}[\mathscr{V}]^{\text{sub}}_{\Sigma^h},$$

canonically.

(**2.5.3**) Let $(G^\#, X^\#) \subset (G, X)$, $\Sigma^\#$, and $\psi_\Sigma: M^\#_{\Sigma^\#} \to M_\Sigma$ be as in (1.3.3). Then

$$\psi^*_\Sigma([\mathscr{V}]^{\text{can}}_{\Sigma}) \cong (\psi^*[\mathscr{V}])^{\text{can}}_{\Sigma^\#}, \qquad \psi^*_\Sigma([\mathscr{V}]^{\text{sub}}_{\Sigma}) \cong (\psi^*[\mathscr{V}])^{\text{sub}}_{\Sigma^\#},$$

canonically.

These facts are proved for canonical extensions in 4.3 of [27]; the case of subcanonical extensions is just as easy.

More canonically, we define

$$H_K^*([\mathscr{V}]^{\mathrm{can}}) = \varinjlim_{\Sigma} H^*({}_K M_{\Sigma, K}[\mathscr{V}]_{\Sigma}^{\mathrm{can}}),$$
$$H_K^*([\mathscr{V}]^{\mathrm{sub}}) = \varinjlim_{\Sigma} H^*({}_K M_{\Sigma, K}[\mathscr{V}]_{\Sigma}^{\mathrm{sub}}),$$

where the K-admissible Σ forms an inverse system under the relation of refinement. The natural homomorphisms

$$H^*({}_K M_{\Sigma, K}[\mathscr{V}]_{\Sigma}^{\mathrm{can}}) \to H_K^*([\mathscr{V}]^{\mathrm{can}}),$$
$$H^*({}_K M_{\Sigma, K}[\mathscr{V}]_{\Sigma}^{\mathrm{sub}}) \to H_K^*([\mathscr{V}]^{\mathrm{sub}})$$

are isomorphisms for any Σ by Proposition (2.4).

The morphisms (2.5.1) and (2.5.2) define homomorphisms

$$(2.5.4) \qquad \begin{aligned} t_{K',K}^*&\colon H_K^*([\mathscr{V}]^{\mathrm{can}}) \to H_{K'}^*([\mathscr{V}]^{\mathrm{can}}),\\ t_{K',K}^*&\colon H_K^*([\mathscr{V}]^{\mathrm{sub}}) \to H_{K'}^*([\mathscr{V}]^{\mathrm{sub}}) \end{aligned}$$

$\forall K' \subset K$, and, for every $h \in G(\mathbf{A}^f)$, and isomorphisms

$$(2.5.5) \qquad \begin{aligned} t_h^*&\colon H_K^*([\mathscr{V}]^{\mathrm{can}}) \xrightarrow{\sim} H_{K^h}^*([\mathscr{V}]^{\mathrm{can}}),\\ t_h^*&\colon H_K^*([\mathscr{V}]^{\mathrm{sub}}) \xrightarrow{\sim} H_{K^h}^*([\mathscr{V}]^{\mathrm{sub}}). \end{aligned}$$

Let $H_K^*([\mathscr{V}])(\infty) = \varinjlim_{\Sigma} H^*({}_K M_\Sigma, [\mathscr{V}]_{Z_\Sigma})$. It follows from Proposition (2.4) and (2.2.4) that the natural map $H^*({}_K M_\Sigma, [\mathscr{V}]_{Z_\Sigma}) \to H_K^*([\mathscr{V}])(\infty)$ is an isomorphism for all Σ. As in (2.5.4) and (2.5.5), we have maps

$$(2.5.6) \qquad \begin{aligned} t_{K',K}^*&\colon H_K^*([\mathscr{V}])(\infty) \to H_{K'}^*([\mathscr{V}])(\infty),\\ t_h^*&\colon H_K^*([\mathscr{V}])(\infty) \xrightarrow{\sim} H_{K^h}^*([\mathscr{V}])(\infty). \end{aligned}$$

Let

$$\tilde{H}^q([\mathscr{V}]^{\mathrm{can}}) = \tilde{H}^q(M(G, X), [\mathscr{V}]^{\mathrm{can}}) = \varinjlim_{K} H_K^q([\mathscr{V}]^{\mathrm{can}}),$$

and define $\tilde{H}^q([\mathscr{V}]^{\mathrm{sub}})$ and $\tilde{H}^q([\mathscr{V}])(\infty)$ analogously, the limits being taken with respect to the homomorphisms $t_{K',K}$.

(2.6) Proposition. *The short exact sequence* (2.2.4) *gives rise, in the limit, to a long exact sequence*

$$(2.6.1) \qquad \cdots \to \tilde{H}^q([\mathscr{V}]^{\mathrm{sub}}) \to \tilde{H}^q([\mathscr{V}]^{\mathrm{can}}) \to \tilde{H}^q([\mathscr{V}])(\infty) \to \tilde{H}^{q+1}([\mathscr{V}]^{\mathrm{sub}}) \to \cdots$$

of admissible $G(\mathbf{A}^f)$*-modules. The long exact sequence* (2.6.1) *is functorial with respect to morphisms* $(G^{\#}, X^{\#}) \to (G, X)$ *of basic pairs.*

Proof. The exactness of the long exact sequence (2.6.1) follows from the exactness of $\varinjlim$. We describe the $G(\mathbf{A}^f)$-actions on the terms of the sequence. Write $V = \tilde{H}^q([\mathscr{V}]^{\mathrm{can}})$. Let $v \in V$, and suppose v is in the image of $H^q({}_KM_{\Sigma,K}[\mathscr{V}]_{\Sigma}^{\mathrm{can}})$. Then $t_h^*(v) \in H^q({}_{K^h}M_{\Sigma^h,K^h}[\mathscr{V}]_{\Sigma^h}^{\mathrm{can}})$, and we let $\pi(h)v$ be the image of $t_h^*(v)$ in V under the natural map. The definitions are analogous for $\tilde{H}^q([\mathscr{V}]^{\mathrm{sub}})$ and $\tilde{H}^q([\mathscr{V}])(\infty)$. One checks easily that, in each case, the association $h \mapsto \pi(h)$ is well defined and a representation. Moreover, it is evidently smooth in the sense that every vector v is stabilized by an open subgroup of $G(\mathbf{A}^f)$. It thus remains to show that, for any open subgroup of $G(\mathbf{A}^f)$, the subspace V^K of K-fixed vectors is finite dimensional, where $V = \tilde{H}^q([\mathscr{V}]^{\mathrm{can}})$, $\tilde{H}^q([\mathscr{V}]^{\mathrm{sub}})$, or $\tilde{H}^q([\mathscr{V}])(\infty)$.

We verify this for $V = \tilde{H}^q([\mathscr{V}]^{\mathrm{can}})$; the other cases are analogous. It suffices to show that V^K is the image of $\tilde{H}_K^q([\mathscr{V}]^{\mathrm{can}})$ in V. Indeed, $\tilde{H}_K^q([\mathscr{V}]^{\mathrm{can}})$ is isomorphic to any $H^q({}_KM_{\Sigma,K}[\mathscr{V}]_{\Sigma}^{\mathrm{can}})$; the assertion is thus a consequence of the finite dimensionality of cohomology of coherent sheaves over complete algebraic spaces [32].

Thus suppose $v \in V^K$. We may assume v to be in the image of $H^q({}_{K'}M_{\Sigma,K'}[\mathscr{V}]_{\Sigma}^{\mathrm{can}})$ for some open normal subgroup $K' \subset K$, where Σ is K-admissible (cf. (1.3.1)). Let $H = K'/K$; we have $v \in H^q({}_{K'}M_{\Sigma,K'}[\mathscr{V}]_{\Sigma}^{\mathrm{can}})^H$. By [27, Lemma 2.6], the quotient ${}_{K'}M_{\Sigma}/H$ exists and is canonically isomorphic to ${}_KM_{\Sigma}$. Now H is finite and ${}_{K'}[\mathscr{V}]_{\Sigma}^{\mathrm{can}}$ is an H-equivariant sheaf (via the morphisms t_h^*) in $\mathbb{Q}$-vector spaces. We may thus identify v with an element of the quotient $H^q({}_KM_{\Sigma}, \mathscr{E})$, where $\mathscr{E}$ is a sheaf over ${}_KM_{\Sigma}$ whose pullback to ${}_{K'}M_{\Sigma}$ is ${}_{K'}[\mathscr{V}]_{\Sigma}^{\mathrm{can}}$. It follows from (2.2.1) that $\mathscr{E}$ is isomorphic to ${}_K[\mathscr{V}]_{\Sigma}^{\mathrm{can}}$. Thus v is in the image of $\tilde{H}_K^q([\mathscr{V}]^{\mathrm{can}})$, and the proposition is proved.

(**2.7**) Now suppose Σ is projective and equivariant, and let L be the field of definition of the homogeneous vector bundle $\mathscr{V}/\check{M}$. Then, by Proposition (2.2.2) and the remarks following (2.2.4), the exact sequence (2.2.4) is L-rational. It follows immediately that the long exact sequence

$$(2.7.1) \qquad \cdots \to H^q({}_KM_{\Sigma,K}[\mathscr{V}]_{\Sigma}^{\mathrm{sub}}) \to H^q({}_KM_{\Sigma,K}[\mathscr{V}]_{\Sigma}^{\mathrm{can}}) \to H^q({}_KM_{\Sigma}, [\mathscr{V}]_{Z_{\Sigma}}) \to \cdots$$

has a natural L-rational structure, and thus defines an L-rational structure on the long exact sequence

$$(2.7.2) \qquad \cdots \to H^q_K([\mathscr{V}]^{\text{sub}}) \to H^q_K([\mathscr{V}]^{\text{can}}) \to H^q_K([\mathscr{V}])(\infty) \to \cdots .$$

If Σ' is another projective equivariant K-admissible collection of polyhedra, then there exists a projective equivariant K-admissible collection of polyhedra Σ'' which is simultaneously a refinement of Σ and Σ'. The morphisms $\pi_{\Sigma'',\Sigma}: {}_K M_{\Sigma''} \to {}_K M_{\Sigma}$ and $\pi_{\Sigma'',\Sigma'}: {}_K M_{\Sigma''} \to {}_K M_{\Sigma'}$ are both rational over $E(G, X)$. It follows that the L-rational structure on (2.7.2) is independent of the choice of projective equivariant Σ.

Moreover, as explained in [27, 4.3.5], the isomorphisms (2.5.1) and (2.5.2) (resp. (2.5.3)) are rational over L (resp. $L \cdot E(G^{\#}, X^{\#})$). Thus we have

(2.8) Proposition. *The long exact sequence* (2.6.1) *of admissible $G(\mathbf{A}^f)$-modules is naturally defined over the field of definition of $\mathscr{V}$ as a G-homogeneous vector bundle over $\check{M}$. Let L be this field of definition. If $(G^{\#}, X^{\#}) \to (G, X)$ is a morphism of basic pairs, then the corresponding map of the long exact sequences is rational over $L \cdot E(G^{\#}, X^{\#})$.*

(2.9) Remark. The methods of Milne [35] imply a stronger version of the above proposition, in the setting of the Langlands conjecture.

3. Applications of the Dolbeault complex with logarithmic singularities

(3.1) The paper of Harris-Phong [28] was designed for applications to the computation of the cohomology groups of the vector bundles $[\mathscr{V}]^{\text{can}}$. In the first part of this section, we work out the modifications of the theory of [28] necessary for application to $[\mathscr{V}]^{\text{can}}$.

Following [28], we let Δ be the disc of radius $\frac{1}{2}$ in $\mathbb{C}$, and let $\Delta^* = \Delta - \{0\}$ be the punctured disc. Let z be the variable in Δ, $r = |z|$. The following lemma is an amplification of Lemme 1 of [28]:

(3.1.1) Lemma. *Let $g \in C^{\infty}(\Delta^*)$ be a function satisfying an estimate of the form*

$$|z \cdot g(z)| < C|\log r|^N, \qquad C \in \mathbb{R}^+, \ N \in \mathbb{Z}.$$

Then the equation $\bar\partial f = g$ has a solution $f \in C^{\infty}(\Delta^*)$ satisfying

$$(3.1.2) \qquad |f(z)| < C'|\log r|^{N+2} \quad \textit{for some } C' \in \mathbb{R}^+.$$

In [28], the exponent in (3.1.2) was not specified. The proof in [28] proceeds by assuming $N \geq 0$ (the article contains the misprint $N \leq 0$) and constructing a solution f of the equation $\bar\partial f = g$ satisfying a certain estimate [28, pp. 308–309]. Whether or not $N \geq 0$, the argument in [28] shows that f satisfies

$$(*)\quad |f(z)| < \frac{4C}{r}\int_0^{r/2} |\log\rho|^N \, d\rho + C^* \left|\log\frac{r}{2}\right|^N + 6C\int_{r/2}^{1/2} \frac{|\log\rho|^N}{\rho}\, d\rho.$$

Now the last term in $(*)$ is $O(|\log\rho|^{N+1})$ except when $N = -1$, in which case it is $o(|\log\rho|^{N+2})$. The first term can be computed by integration by parts for $N \geq 0$ (again, there is a misprint in the article, but the final result is correct); we obtain $4CP_N(|\log(r/2)|)$, where P_N is a polynomial of degree N. For $N < 0$ this is no longer possible. However, in that case, the function $|\log\rho|^N$ is *increasing* on the interval $(0, r/2)$. We thus have

$$\frac{4C}{r}\int_0^{r/2} |\log\rho|^N \, d\rho \leq \frac{4C}{r}\cdot\left|\log\frac{r}{2}\right|^N \cdot \frac{r}{2} = 2C\left|\log\frac{r}{2}\right|^N.$$

Combining these estimates, we obtain the desired amplification.

We say $f \in C^\infty(\Delta^*)$ is *slowly increasing* (resp. *rapidly decreasing*) near 0 if f satisfies an estimate of the form

$$|f(z)| < C|\log r|^N$$

for some $N \in \mathbb{Z}$ (resp., for all $N \in \mathbb{Z}$). Lemma (3.1.1) implies

(3.1.3) Corollary. *Suppose $g \in C^\infty(\Delta^*)$ is a function such that $|z|\cdot g(z)$ is slowly increasing (resp. rapidly decreasing) near 0. Then the equation $\bar\partial f = g$ has a solution $f \in C^\infty(\Delta^*)$ which is slowly increasing (resp. rapidly decreasing) near 0.*

(3.1.4) As in [28], we let $X = X_{n,r} = (\Delta^*)^r \times \Delta^{n-r} \subset \Delta^n$, $0 \leq r \leq n$, with coordinates $z_1, \cdots, z_n$. We say $f \in C^\infty(X)$ is *slowly increasing* (resp. *rapidly decreasing*) if for any compact $\kappa \subset \Delta^n$, f satisfies an estimate of the form

$$(3.1.4.1)\qquad |f(z)| < C_\kappa \left(\sum_{i=1}^r \log|z_i|\right)^{2N}, \qquad C_\kappa \in \mathbb{R}^+,\ z \in X_{n,r} \cap \kappa,$$

for some $N \in \mathbb{Z}$ depending on κ (resp. for all $N \in \mathbb{Z}$; here C_κ may depend on N). We let $C^\infty_{\mathrm{si}}(X)$ (resp. $C^\infty_{\mathrm{rd}}(X)$) denote the space of C^∞ slowly increasing (resp. rapidly decreasing) functions on X, and let $A^\cdot_*(X)$, where $*$ is either si or rd, denote the algebra over $C^\infty_*(X)$ generated by the differentials $d\bar z_1/|z_1|, \cdots, d\bar z_r/|z_r|, d\bar z_{r+1}, \cdots, d\bar z_n$. Finally, for $* =$ si or rd, let $K^\cdot_*(X)$ be the following subcomplex of the

Dolbeault complex $\Omega^{0,\cdot}(X)$:

$$K^{\cdot}_{*}(X) = \{\omega \in A^{\cdot}_{*}(X) | \bar{\partial}\omega \in A^{\cdot}_{*}(X)\}.$$

(In [28], C^{∞}_{si} and $K^{\cdot}_{\mathrm{si}}$ were denoted $C^{\infty}_{\log}$ and $K^{\cdot}_{\log}$, respectively.) For each $q \geq 0$, $K^{q}_{*}(X)$ is a module over the algebra of C^{∞} functions on Δ^n with compact support. As in [28], we have

(3.1.4.2) Corollary (*to* (3.1.3)). *The cohomology groups* $H^i(K^{\cdot}_{*}(X))$ *are trivial for* $i > 0$.

(3.1.4.3) Lemma. *The space* $H^0(K^{\cdot}_{\mathrm{si}}(X))$ (*resp.* $H^0(K^{\cdot}_{\mathrm{rd}}(X))$) *is equal to the space of holomorphic functions on* Δ^n (*resp. to the space of holomorphic functions on* Δ^n *which vanish on the closed subset* $\Delta^n - X$).

Proof. The case of $K^{\cdot}_{\mathrm{si}}(X)$ is Lemma 2 of [28]. It follows that $H^0(K^{\cdot}_{\mathrm{rd}}(X))$ is the space of holomorphic functions on Δ^n which are rapidly decreasing on X, from which the lemma follows immediately.

(3.1.5) Let V be a smooth algebraic variety over $\mathbb{C}$ of dimension n, and let $Z \subset V$ be a divisor with normal crossings. Let $j \colon U =_{\mathrm{def}} V - Z \hookrightarrow V$ be the inclusion. Every point $x \in V$ has a neighborhood D_x which admits an analytic chart

$$\text{(3.1.5.1)} \qquad \varphi \colon (U \cap D_x, D_x, x) \xrightarrow{\sim} (X_{n,r}, \Delta^n, 0);$$

such a φ is said to be *admissible* if it extends to an isomorphism between a neighborhood of the closure of D_x in V and an open polydisc containing the closure of Δ^n in $\mathbb{C}^n$.

Following [28], we define C^{∞} sheaves on V:

$$\Omega^{0,\cdot}_V \subset \mathscr{K}^{\cdot}_{\mathrm{rd}} \subset \mathscr{K}^{\cdot}_{\mathrm{si}} \subset j_*\Omega^{0,\cdot}_U:$$

$\mathscr{K}^{q}_{*}$, where $* = \mathrm{si}$ or rd, is the subsheaf of $j_*\Omega^{0,q}_U$ whose sections over the open set D_x are contained in $C^{\infty}(\Delta^n) \otimes K^{q}_{*}(X_{n,r})$, relative to any admissible chart φ.

Now if $\mathscr{E}$ is any holomorphic vector bundle (or locally free sheaf) over V, we let $\mathscr{K}^{\cdot}_{*,Z}(\mathscr{E})$ be the complex $\mathscr{E} \otimes \mathscr{K}^{\cdot}_{*}$, for $* = \mathrm{si}$ or rd, with differential $1 \otimes \bar{\partial}$. As usual, $\mathscr{O}_V(-Z)$ denotes the subsheaf of $\mathscr{O}_V$ of functions vanishing along the divisor Z; $\mathscr{E}(-Z) = \mathscr{E} \otimes \mathscr{O}_V(-Z)$. The following amplification of the theorem in [28] follows from the arguments there and from (3.1.4.2) and (3.1.4.3):

(3.1.6) Theorem. *The complex* $\mathscr{K}^{\cdot}_{\mathrm{rd},Z}(\mathscr{E})$ (*resp.* $\mathscr{K}^{\cdot}_{\mathrm{si},Z}(\mathscr{E})$) *is a fine resolution of the sheaf* $\mathscr{E}(-Z)$ (*resp. of the sheaf* $\mathscr{E}$). *In particular, we have a natural commutative diagram, where the horizontal arrows are isomorphisms:*

$$\begin{array}{ccc} H^q(V, \mathscr{E}(-Z)) & \xrightarrow{\sim} & H^q_{\bar\partial}(\mathscr{K}^{\cdot}_{\mathrm{rd},Z}(\mathscr{E})) \\ \downarrow & & \downarrow \\ H^q(V, \mathscr{E}) & \xrightarrow{\sim} & H^q_{\bar\partial}(\mathscr{K}^{\cdot}_{\mathrm{si},Z}(\mathscr{E})) \end{array}$$

Here the left vertical arrow comes from the inclusion of sheaves $\mathscr{E}(-Z) \subset \mathscr{E}$.

(**3.2**) We return to the notation of (2.0). Fix a level subgroup $K \subset G(\mathbf{A}^f)$. Let $\sigma: K_\infty \to \mathrm{GL}(V_\sigma)$ be a finite-dimensional representation, and let $[\mathscr{V}] = [\check{E}_\sigma]$ be the corresponding fully decomposed automorphic vector bundle over $M = {}_KM(G, X)$. We view M as an analytic quotient space of $X \times G(\mathbf{A}^f)$. Recall that the pullback of $[\mathscr{V}]$ to $X \times G(\mathbf{A}^f)$ is a $G(\mathbf{A})$-equivariant holomorphic vector bundle $\mathscr{V}^0$. Let $\mathscr{V}^0_h$ be the fiber of $\mathscr{V}^0$ at h.

The intersection $K_{\infty,0} = K_\infty \cap G_0$ is compact. Thus the representation $\sigma: K_{\infty,0} \to \mathrm{GL}(\mathscr{V}^0_h)$ fixes a positive-definite Hermitian form on $\mathscr{V}^0_h$, and thus defines a $G_0 \times G(\mathbf{A}^f)$-invariant Hermitian metric $\tilde{h}(\cdot,\cdot) = \tilde{h}_\sigma(\cdot,\cdot)$ on $\mathscr{V}^0$. Let $h(\cdot,\cdot) = h_{\mathscr{V}}(\cdot,\cdot)$ be the corresponding Hermitian metric on $[\mathscr{V}]$ over M.

Fix a projective SNC toroidal compactification $j: M \hookrightarrow \bar{M} = M_\Sigma$. Let $[\mathscr{V}]^{\mathrm{can}}$ and $[\mathscr{V}]^{\mathrm{sub}}$ be the canonical and subcanonical extensions of E over $\bar{M}$. We apply the theory of (3.1) to $\mathscr{E} = [\mathscr{V}]^{\mathrm{can}}$. Let $n = \dim X$. Every point $m \in \bar{M}$ has a neighborhood $D_m \subset \bar{M}$ which admits an admissible analytic isomorphism $\varphi: (X_{n,r}, \Delta^n, 0) \xrightarrow{\sim} (M \cap D_m, D_m, m)$ as in (3.1.5.1). Let $\{e_1, \cdots, e_d\}$ be a basis of sections of $[\mathscr{V}]^{\mathrm{can}}$ over a neighborhood of the closure of D_m in $\bar{M}$. We may consider the values $h_{i,j} =_{\mathrm{def}} h_{\mathscr{V}}(e_i, e_j)$, $1 \le i, j \le d$, as functions of $\varphi(z)$ on $\bar{M} \cap D_m$, and thus as functions on $X_{n,r}$. In [36], Mumford proves that

(3.2.1) $\qquad |h_{ij}(\varphi(z))|$ is slowly increasing $\forall i, j$.

Let $s(\cdot,\cdot)$ be a Hermitian metric on ${}_KM$ which pulls back to a $G(\mathbf{A})$-invariant Hermitian metric on $X \times G(\mathbf{A}^f)$. Let $\Omega^{0,q}_M$ be the sheaf of C^∞ differential forms of type $(0, q)$ on M, $q = 0, \cdots, n$. For each q, $s(\cdot,\cdot)$ induces a Hermitian metric $h^q(\cdot,\cdot)$ on $\Omega^{0,q}_M$. Recall that $\mathscr{K}^{\cdot}_{\mathrm{si}}$ and $\mathscr{K}^{\cdot}_{\mathrm{rd}}$, as defined above, are subcomplexes of $j_*\Omega^{0,\cdot}_M$. It follows from [36, Proposition 3.4 and Theorem 3.1] that, given two sections $s_1, s_2 \in \Gamma(D_m, \mathscr{K}^q_{\mathrm{si}})$ (resp. $\Gamma(D_m, \mathscr{K}^q_{\mathrm{rd}})$), (3.2.2) $|h^q(s_1, s_2)(\varphi(z))|$ is slowly increasing (resp. rapidly decreasing).

Combining (3.2.1) and (3.2.2), we have

(3.2.3) Lemma. *Let* m, D_m, r, *and* φ *be as above. Then* $\Gamma(D_m, \mathscr{K}^q_{\mathrm{si},Z}([\mathscr{V}]^{\mathrm{can}}))$ *(resp.* $\Gamma(D_m, \mathscr{K}^q_{\mathrm{rd},Z}([\mathscr{V}]^{\mathrm{sub}})))$ $\subset \Gamma(D_m \cap M, [\mathscr{V}]\otimes \Omega^{0,q}_M)$ *consists of the* $[\mathscr{V}]$*-valued* $C^\infty(0,q)$ *forms* s *on* $D_m \cap M \cong X_{n,r}$ *with the property that* $|h^q_{\mathscr{V}}(s, s)(\varphi(z))|$ *and* $|h^{q+1}_{\mathscr{V}}(\bar\partial s, \bar\partial s)(\varphi(z))|$ *are slowly increasing (resp. rapidly decreasing).*

Proof. This follows easily from the estimates (3.2.1) and (3.2.2) by a Gram-Schmidt argument.

(3.3) We now lift our $[\mathscr{V}]$-valued $(0,q)$-forms to the adele group. Define $\mathfrak{p}^+$ and $\mathfrak{p}^-$ as in (2.0), and let $\alpha: X \times G(\mathbf{A}^f) \to M$ be the natural map. The fiber of $[\mathscr{V}]\otimes\Omega^{0,q}_M$ at $\alpha(h\times 1)$ is naturally isomorphic to $V_\sigma \otimes \Lambda^q(\mathfrak{p}^-)^* \cong V_\sigma \otimes \Lambda^q \mathfrak{p}^+$ as K_∞-modules; denote this K_∞-module by V^q_σ, and let $\sigma^q: K_\infty \to \mathrm{GL}(V^q_\sigma)$ be the given representation, for $q = 0, \cdots, n$. Let $\beta: G(\mathbf{A}) \to X \times G(\mathbf{A}^f)$ be the natural map. Then $\beta^*\alpha^*([\mathscr{V}]\otimes\Omega^{0,q}_M)$ is canonically isomorphic to $G(\mathbf{A}) \times V^q_\sigma$. There is thus a canonical isomorphism

(3.3.1)

$$\underline{\mathrm{Lift}}: \Gamma(M, [\mathscr{V}]\otimes\Omega^{0,q}_M) \xrightarrow{\sim} \{f \in C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})/K, V^q_\sigma) | f(gk) = \sigma^q(k)^{-1} f(g) \quad \forall g \in G(\mathbf{A}),\ k \in K_\infty\}.$$

We denote the right-hand side of (3.3.1) by ${}_K C^q_\sigma$. Note that we have a canonical identification

$${}_K C^q_\sigma = \mathrm{Hom}_{K_\infty}((V^q_\sigma)^*, C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})/K)) \subset C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})/K)\otimes V^q_\sigma,$$

where K_∞ acts on $C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))$ by right translation.

Let $C^q_\sigma = \varinjlim_K {}_K C^q_\sigma$, $K \subset G(\mathbf{A}^f)$ an open compact subgroup. For each $q = 0, \cdots, n$, and $* = \mathrm{si}$ or rd, let

$$\mathscr{A}^q_*([\mathscr{V}]) = \varinjlim_{K,\Sigma} \Gamma({}_K M_\Sigma, \mathscr{K}^q_{*,Z_\Sigma}([\mathscr{V}]^{\mathrm{can}})).$$

In the limit, (3.3.1) defines a canonical lifting $\underline{\mathrm{Lift}}: \mathscr{A}^q_*([\mathscr{V}]) \hookrightarrow C^q_\sigma$. The image of $\mathscr{A}^q_*([\mathscr{V}])$ under $\underline{\mathrm{Lift}}$ is denoted $C^q_{*,\sigma}$, for $* = \mathrm{si}$ or rd. The differential $\bar\partial: \mathscr{A}^q_*([\mathscr{V}]) \to \mathscr{A}^{q+1}_*([\mathscr{V}])$ induces a homomorphism, also denoted $\bar\partial: C^q_{*,\sigma} \to C^{q+1}_{*,\sigma}$.

Following Mumford [36] again, we can translate the growth condition in Lemma (3.2.3) into a condition on $\underline{\mathrm{Lift}}(s)$. Let $g \mapsto \tilde g$ be the Cartan involution on G_0 with respect to K_∞, and define $||g||_{G_0} = \mathrm{tr}\,\mathrm{Ad}(\tilde g^{-1}\cdot g)$,

$g \in G_0$, as in Borel's article [6]. If $(V, \|\ \|_V)$ is a normed complex vector space, $f \in C^\infty(G(\mathbf{A}), V)$ is called *slowly increasing* (resp. *rapidly decreasing*) if

(3.3.2) $\qquad f$ is a finite sum of eigenfunctions for $Z_G(\mathbf{A})$,

(3.3.3)
$$\|f(g_0\gamma)\|_V < C\|g_0\|_{G_0}^n$$
$$\forall g_0 \in C_0,\ \gamma \in G(\mathbf{A}),\ \text{for some (resp. for all)}$$
$$n \geq 0,\ C \in \mathbb{R}^+\ (\text{resp. } C \in \mathbb{R}^+ \text{ depending on } n \text{ and } \gamma).$$

The condition (3.3.2), which is not traditional, is automatic for $f \in C^q_\sigma$, because $Z_G(\mathbb{R}) \subset K_\infty$ and $Z_G(\mathbb{Q}) \cdot Z_G(\mathbb{R}) \backslash Z_G(\mathbf{A})$ is compact.

(3.3.4) Lemma. *The space $C^q_{\mathrm{si},\sigma}$ (resp. $C^q_{\mathrm{rd},\sigma}$) is the subspace of functions $f \in C^q_\sigma$ such that both f and $\bar\partial f$ are slowly increasing (resp. rapidly decreasing) on $G(\mathbf{A})$, in the above sense, where $V = V^q_\sigma$ or V^{q+1}_σ, as the case may be.*

Proof. An equivalent statement is proved by Mumford as Proposition 3.3 of [36].

We now regard $[\mathscr{V}]$ as an automorphic vector bundle over $M(G, X)$, and define $\tilde{H}^*([\mathscr{V}]^{\mathrm{sub}})$ and $\tilde{H}^*([\mathscr{V}]^{\mathrm{can}})$ as in Proposition (2.6). Combining the above lemmas with Theorem (3.1.6), we obtain

(3.4) Corollary. *Let $[\mathscr{V}] = [\check{E}_\sigma]$ be the fully decomposed automorphic vector bundle corresponding to the representation σ of K_∞. There is a natural commutative diagram of admissible $G(\mathbf{A}^f)$ modules:*

(3.4.1)
$$\begin{array}{ccc} H^*_{\bar\partial}(C^\cdot_{\mathrm{rd},\sigma}) & \xrightarrow{\ \sim\ } & \tilde{H}^*([\mathscr{V}]^{\mathrm{sub}}) \\ \downarrow & & \downarrow \\ H^*_{\bar\partial}(C^\cdot_{\mathrm{si},\sigma}) & \xrightarrow{\ \sim\ } & \tilde{H}^*([\mathscr{V}]^{\mathrm{can}}) \end{array}$$

The horizontal arrows are isomorphisms.

Proof. We have proved everything except that the horizontal arrows commute with the $G(\mathbf{A}^f)$-actions on the two sides; but this follows trivially from the obvious functorial properties of the complexes $\mathscr{K}^\cdot_{\mathrm{rd},Z}$ and $\mathscr{K}^\cdot_{\mathrm{si},Z}$ with respect to the morphisms t_h and $t_{K',K}$ of (1.3).

(3.4.2) Remark. The above arguments provide another proof of Proposition (2.4), when M_Σ and $M_{\Sigma'}$ are both assumed to be SNC. However, our original algebraic proof is necessary for the rationality statements of Proposition (2.8).

(3.5) In the remainder of this section, we apply some ideas of Borel [8], [9] to study the image of the harmonic cusp forms in $\tilde{H}^*([\mathscr{V}]^{\text{can}})$ under the horizontal map in (3.4.1).

We begin with some notation. Fix a level subgroup K, and let $M = {}_K M(G, X)$. Let $s(\cdot,\cdot)$ be the Hermitian metric introduced in (3.2); we note that M is complete with respect to $s(\cdot,\cdot)$. Let $\Omega^{p,q}([\mathscr{V}])$ denote the bundle of $C^\infty(p, q)$-forms on M with coefficients in $[\mathscr{V}]$, $\mathscr{A}^{p,q}([\mathscr{V}]) = \Gamma(M, \Omega^{p,q}([\mathscr{V}]))$ and let $\bar\partial_{\mathscr{V}} = \bar\partial : \mathscr{A}^{p,q}([\mathscr{V}]) \to \mathscr{A}^{p,q+1}([\mathscr{V}])$. The Hermitian metric $h_{\mathscr{V}}$ on $[\mathscr{V}]$, together with s, defines a metric $h_{\mathscr{V}}^{p,q}$ on $\Omega^{p,q}([\mathscr{V}])$ and a star operator

$$*_{\mathscr{V}} : \mathscr{A}^{p,q}([\mathscr{V}]) \to \mathscr{A}^{n-p,n-q}([\mathscr{V}])^*).$$

We let

$$\delta_{\mathscr{V}} = - * \bar\partial_{\mathscr{V}*} : \mathscr{A}^{p,q+1}([\mathscr{V}]) \to \mathscr{A}^{p,q}([\mathscr{V}])$$

be the formal adjoint to $\bar\partial$.

If $\alpha \in \mathscr{A}^{p,q}([\mathscr{V}])$, let $|\alpha| = |\alpha|^{p,q} = h_{\mathscr{V}}^{p,q}(\alpha, \alpha)^{1/2}$, as a function on M. We write α_x for the value of α at x. For $\alpha, \beta \in \mathscr{A}^{p,q}([\mathscr{V}])$, we let

$$(\alpha, \beta)_M = \int_M [\alpha \wedge *_{\mathscr{V}} \beta], \tag{3.5.1}$$

where $[\cdot]$ is the contraction $\mathscr{A}^{n,n}([\mathscr{V}]) \otimes [\mathscr{V}]^*) \to \mathscr{A}^{n,n}(\mathscr{O}_M)$, whenever the integral is defined.

The following proposition is a special case of Proposition 2.2 of [8]:

(3.5.2) Proposition (*Borel* [8]). *Let $\alpha \in \mathscr{A}^{p,q-1}([\mathscr{V}])$ and $\beta \in \mathscr{A}^{p,q}([\mathscr{V}])$. Assume that the functions $x \mapsto |\alpha_x|\cdot|\beta_x|$, $x \mapsto h_{\mathscr{V}}^{p,q}((\bar\partial_{\mathscr{V}}\alpha)_x, \beta_x)$, and $x \mapsto h_{\mathscr{V}}^{p,q-1}(\alpha_x, (\delta_{\mathscr{V}}\beta)_x)$ are integrable on M. Then*

$$(\alpha, \delta_{\mathscr{V}}\beta)_M = (\bar\partial_{\mathscr{V}}\alpha, \beta)_M. \tag{3.5.3}$$

Proof. Borel's proposition is actually stated only for M a complete Riemannian manifold and $[\mathscr{V}]$ the trivial bundle. However, since M is complete, it is known [1, p. 88] that (3.5.3) holds whenever one of α, β has compact support. The proposition now follows from Borel's argument in [8].

(3.5.4) We write $\mathscr{A}^q([\mathscr{V}]) = \mathscr{A}^{0,q}([\mathscr{V}])$. Define the Laplace-Beltrami operator

$$\Box_{q,\sigma} = \bar\partial_{\mathscr{V}}\delta_{\mathscr{V}} + \delta_{\mathscr{V}}\bar\partial_{\mathscr{V}} : \mathscr{A}^q([\mathscr{V}]) \to \mathscr{A}^q([\mathscr{V}]).$$

Applying Lift (3.3.1), we obtain an operator

$$\Box_{q,\sigma} : {}_K C_\sigma^q \to {}_K C_\sigma^q, \qquad q = 0, \cdots, n.$$

It is well known [1] that, if $\alpha \in \mathscr{A}^q([\mathscr{V}])$, $(\alpha, \alpha)_M < \infty$, then

$$(3.5.5) \qquad \square_{q,\sigma}(\alpha) = 0 \Leftrightarrow \bar{\partial}_{\mathscr{V}}(\alpha) = 0 \quad \text{and} \quad \delta_{\mathscr{V}}(\alpha) = 0.$$

Now let $\varphi \in {}_KC^q_{\mathrm{si},\sigma} = {}_KC^q_\sigma \cap C^q_{\mathrm{si},\sigma}$ be a harmonic cusp form, $\varphi = \underline{\mathrm{Lift}}(\alpha)$, $\alpha \in \mathscr{A}^q([\mathscr{V}])$. In other words, we assume the following:

(i) φ is an automorphism form, which under the hypothesis reduces to saying that φ is annihilated by an ideal of finite codimension in $Z(\mathfrak{g}_{\mathbf{C}})$;

(ii) φ is a cusp form, i.e., the constant term of the Fourier expansion of φ along $U(\mathbf{A})$ vanishes whenever U is the unipotent radical of a rational parabolic subgroup of G [11];

(iii) $\square_{q,\sigma}(\varphi) = 0$.

It follows already from (i) and (ii) that $\varphi \in {}_KC^q_{\mathrm{rd},\sigma}$. Thus

(3.5.6) The functions $x \mapsto h^{0,q}_{\mathscr{V}}(\alpha_x, \beta_x)$ and $x \mapsto |\alpha_x|^{0,q} \cdot |\gamma_x|^{0,q'}$ are integrable on M $\forall \beta \in \mathscr{A}^q_{\mathrm{si}}(E)$, $\gamma \in \mathscr{A}^{q'}_{\mathrm{si}}(E)$, $q' = 0, \cdots, n$.

Let ${}_K\mathscr{H}^q_{\mathrm{cusp}} = {}_K\mathscr{H}^q_{\mathrm{cusp},\sigma}$ be the space of harmonic cusp forms in ${}_KC^q_{\mathrm{rd},\sigma}$. It follows from (iii) and (3.5.5) that $\bar{\partial}\varphi = 0$ $\forall \varphi \in {}_K\mathscr{H}^q_{\mathrm{cusp}}$. There is thus a homomorphism

$$(3.5.7) \qquad s\colon {}_K\mathscr{H}^q_{\mathrm{cusp}} \to H^q_{\bar{\partial}}({}_KC^{\cdot}_{\mathrm{rd},\sigma}) \cong K^q_K([\mathscr{V}]^{\mathrm{sub}}),$$

in the notation of (2.5); the last isomorphism is given by Corollary (3.4). Let $\bar{H}^q_K([\mathscr{V}])$ denote the image of $H^q_K([\mathscr{V}]^{\mathrm{sub}})$ in $H^q_K([\mathscr{V}]^{\mathrm{can}})$. For $\varphi \in {}_K\mathscr{H}^q_{\mathrm{cusp}}$, we let $\mathrm{cl}(\varphi)$ denote the image of $s(\varphi)$ in $\bar{H}^q_K([\mathscr{V}])$.

Let $\mathscr{H}^q_{\mathrm{cusp}} = \mathscr{H}^q_{\mathrm{cusp},\sigma} = \varinjlim_K {}_K\mathscr{H}^q_{\mathrm{cusp}} \subset C^q_{\mathrm{rd},\sigma}$, and let $\bar{H}^q([\mathscr{V}]) = \varinjlim_K \bar{H}^q_K([\mathscr{V}]) =$ the image of $\tilde{H}^q([\mathscr{V}]^{\mathrm{sub}})$ in $\tilde{H}^q([\mathscr{V}]^{\mathrm{can}})$. Let $\mathrm{cl}\colon \mathscr{H}^q_{\mathrm{cusp}} \to \bar{H}^q([\mathscr{V}])$ denote the direct limit of the homomorphisms defined in the previous paragraph.

(3.6) Proposition. *The homomorphism* $\mathrm{cl}\colon \mathscr{H}^q_{\mathrm{cusp}} \to \bar{H}^q([\mathscr{V}])$ *is an injective homomorphism of admissible* $G(\mathbf{A}^f)$*-modules.*

Proof. By Corollary (3.4), it is enough to show that, for $0 \neq \varphi \in {}_K\mathscr{H}^q_{\mathrm{cusp}}$, the class defined by $s(\varphi)$ in $H^q_{\bar{\partial}}({}_KC^{\cdot}_{\mathrm{si},\sigma}) = H^q_{\bar{\partial}}(\mathscr{A}^{\cdot}_{\mathrm{si}}([\mathscr{V}]))$ is nontrivial. Thus suppose $\varphi = \mathrm{Lift}(s)$, $s \in \mathscr{A}^q_{\mathrm{rd}}([\mathscr{V}])$, and suppose $s = \bar{\partial}_{\mathscr{V}} s'$, for some $s' \in \mathscr{A}^{q-1}_{\mathrm{si},\sigma}([\mathscr{V}])$. Note that $\delta_{\mathscr{V}} s = 0$ (3.5.5). It then follows from (3.5.6) that the hypotheses of (3.5.2) are satisfied, with $\alpha = s'$, $\beta = s$, and $p = 0$. Thus

$$(3.6.1) \qquad (s, s)_M = (\bar{\partial}_{\mathscr{V}} s', s)_M = (s', \delta_{\mathscr{V}} s) = 0.$$

But if $G(\mathbb{Q})_0 = G(\mathbb{Q}) \cap G_0$, then

$$(3.6.2) \qquad (s, s)_M = \int_{G(\mathbb{Q})_0 \backslash G_0 \times G(\mathbf{A}^f)/K_0 K} (\varphi, \varphi)_{\sigma^q}\, dg$$

for some Haar measure dg on $G(\mathbf{A})$, and some K_0-invariant Hermitian inner product $(\cdot, \cdot)_{\sigma^q}$ on V_σ^q. The right-hand side of (3.6.2) is just the L_2 metric on C_σ^q, so the proposition is clear.

(**3.7**) Finally, Serre duality (Corollary (2.3)) has the expected interpretation in terms of the isomorphisms (3.4.1). Let τ (resp. ω) be the representation of K_∞ corresponding to $[\mathscr{V}]'$ (resp. to $\mathbb{K} = \Omega^n_{{}_K M}$). The natural pairing $[\mathscr{V}] \otimes [\mathscr{V}]' \to \mathbb{K}$ defines a morphism of complexes

$$(3.7.1) \qquad C^\cdot_{\mathrm{rd}, \tau} \otimes C^\cdot_{\mathrm{si}, \sigma} \to C^\cdot_{\mathrm{rd}, \omega}; \quad f \otimes g \mapsto [f \wedge g],$$

where the double complex on the left-hand side is identified with the associated single complex. We let $\mathscr{A}^{n,n}_{\mathrm{rd}}$ (resp. ${}_K\mathscr{A}^{n,n}_{\mathrm{rd}}$) denote the space of rapidly decreasing n, n forms on $G(\mathbb{Q})\backslash G(\mathbf{A})/K_\infty$ (resp. $G(\mathbb{Q})\backslash G(\mathbf{A})/KK_\infty$). The inverse of <u>Lift</u> identifies $C^n_{\mathrm{rd},\omega} \cong \mathscr{A}^{n,n}_{\mathrm{rd}}$; this permits us to define integration of elements of ${}_K C^n_{\mathrm{rd},\omega}$ over $G(\mathbb{Q})\backslash G(\mathbf{A})/KK_\infty$ for any $K \subset G(\mathbf{A}^f)$. (Recall that $Z_G(\mathbb{R}) \subset K_\infty$.)

(3.8) Proposition. *Let $\varphi \in H_K^{n-q}([\mathscr{V}]'^{,\mathrm{sub}})$ and $\psi \in H^q_K([\mathscr{V}]^{\mathrm{can}})$. Let f (resp. g) be a $\bar{\partial}$-closed form in ${}_K C^{n-q}_{\mathrm{rd},\tau}$ (resp. ${}_K C^q_{\mathrm{si},\sigma}$) representing the cohomology class φ (resp. ψ). We write ${}_K M = G(\mathbb{Q})\backslash G(\mathbf{A})/K_\infty K$. Then the Serre duality pairing is given, up to a constant multiple, by*

$$(3.8.1) \qquad \langle \varphi, \psi \rangle = (2\pi i)^{-n} \int_{G(\mathbb{Q})\backslash G(\mathbf{A})/KK_\infty} [f \wedge g].$$

Proof. Since (3.7.1) is a morphism of complexes, $f \wedge g$ represents the class $\varphi \cup \psi \in H^n_K(\mathbb{K}^{\mathrm{sub}})$, where we denote by $\cup$ the pairing

$$H_K^{n-q}([\mathscr{V}]'^{,\mathrm{sub}}) \otimes H^q_K([\mathscr{V}]^{\mathrm{can}}) \to H^n_K(\mathbb{K}^{\mathrm{sub}})$$

of (2.5.6). It thus suffices to show that, for each connected component M^0 of ${}_K M$, the map

$$(3.8.2) \qquad {}_K C^n_{\mathrm{rd},\omega} \to \mathbb{C}; \quad \beta \mapsto \int_{M^0} \beta$$

induces a surjective morphism

$$H^n({}_K C^\cdot_{\mathrm{rd},\omega}) \to \mathbb{C}.$$

Obviously (3.8.2) is nontrivial (take β positive and compactly supported on M^0). We thus have to show that, if α is a rapidly decreasing $(n, n-1)$ form on ${}_KM$ such that $\bar{\partial}\alpha$ is also rapidly decreasing, then

(3.8.3) $$\int_{{}_KM} \bar{\partial}\alpha = 0.$$

But this is just (1) of [9, 5.6].

(**3.8.4**) Henceforward, Serre duality is taken to be normalized by the formula (3.8.1). This is consistent with the rational structure on $H^n_K(\mathbb{K}^{\text{sub}})$ determined by the identification of the latter with $K^n_{\text{DR}}({}_KM)$; cf. [19, I, Proposition 1.5].

(3.8.5) Corollary. *Let $K \subset G(\mathbf{A}^f)$ be a level subgroup. For any q, the homomorphisms $H^{n-q}_K([\mathscr{V}]'^{,\text{sub}}) \to H^{n-q}_K([\mathscr{V}]'^{,\text{can}})$ and $H^q_K([\mathscr{V}]^{\text{sub}}) \to H^q_K([\mathscr{V}]^{\text{can}})$ are dual to one another with respect to Serre duality, which induces an isomorphism $\bar{H}^q_K([\mathscr{V}]) \xrightarrow{\sim} \bar{H}^{n-q}_K([\mathscr{V}]')^*$.*

Proof. This is an immediate consequence of Proposition (3.8).

(3.8.6) Remark. With a little more work, Proposition (3.8) can be proved for an arbitrary pair (V, Z) as in (3.1).

4. Unitary representations with $\bar{\partial}$-cohomology

(**4.1**) We retain the notation of (2.0). In what follows, all $(\mathfrak{g}, K_\infty)$-modules will be assumed to be complex vector spaces. If V is a $\mathfrak{g}$-module on which K_∞ acts, consistently with the adjoint action of K_∞ on $\mathfrak{g}$, we let V_0 denote the space of K_∞-finite vectors in V. Let $\mathfrak{P}_h$ be the parabolic subalgebra $\text{Lie}(\mathscr{P}_h)$ of $\mathfrak{g}_{\mathbb{C}}$.

(4.1.1) Definition. Let (π, V) be a $\mathfrak{g}$-module on which K_∞ acts, consistently with the adjoint action of K_∞ on $\mathfrak{g}$; let (σ, V_σ) be a finite-dimensional representation of K_∞. We say (π, V) *has $\bar{\partial}$-cohomology with coefficients in σ* (or in V_σ) if the relative Lie algebra cohomology $H^*(\mathfrak{P}_h, K_\infty, V \otimes V_\sigma) \neq \{0\}$. If $H^q(\mathfrak{B}_h, K_\infty, V \otimes V_\sigma) \neq \{0\}$ for some q, we say (π, V) *has $\bar{\partial}$-cohomology in degree q with coefficients in σ* (or in V_σ).

We recall briefly the standard construction of the relative Lie algebra cohomology $H^*(\mathfrak{P}_h, K_\infty, V \otimes V_\sigma)$ [12, §1]. Let W be a module over $\mathfrak{P}_h$, and let

(4.1.2)
$$C^q(\mathfrak{P}_h, K_\infty, W) = \text{Hom}_{K_\infty}(\Lambda^q(\mathfrak{P}_h/\mathfrak{k}_{\infty,\mathbb{C}}), W) = \text{Hom}_{K_\infty}(\Lambda^q(\mathfrak{p}^-), W)$$

for $q = 0, 1, \cdots, \dim \mathfrak{p}^-$; define $d: C^q(\mathfrak{P}_h, K_\infty, W) \to C^{q+1}(\mathfrak{P}_h, K_\infty, W)$

by the formula

(4.1.3)

$$df(x_0, \cdots, x_q) = \sum_i (-1)^i x_i \cdot f(x_0, \cdots, \hat{x}_i, \cdots, x_q) + \sum_{i<j} (-1)^{i+1} f([x_i, x_j], x_0, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_q),$$

where $\{x_0, \ldots, x_q\} \subset \mathfrak{p}^-$ and the variables wearing $\wedge$ are omitted from the summation. Then $d^2 = 0$ and $H^*(\mathfrak{P}_h, K_\infty, W) = H^*(C^\cdot(\mathfrak{P}_h, K_\infty, W))$.

In our case, $W = V \otimes V_\sigma$; the action of $\mathfrak{P}_h$ on V is the restriction of the $\mathfrak{g}_\mathbb{C}$-action, whereas the action of $\mathfrak{P}_h$ on V_σ factors through the quotient $\mathfrak{k}_{\infty,\mathbb{C}}$ of $\mathfrak{P}_h$. In particular, if d_V (resp. $d_{V\otimes V_\sigma}$) is the differential in the complex $C^\cdot(\mathfrak{P}_h, K_\infty, V)$ (resp. $C^\cdot(\mathfrak{P}_h, K_\infty, V \otimes V_\sigma)$), then $d_{V\otimes V_\sigma} = d_V \otimes 1$, under the isomorphism $C^\cdot(\mathfrak{P}_h, K_\infty, V \otimes V_\sigma) \cong C^\cdot(\mathfrak{P}_h, K_\infty, V) \otimes V_\sigma$.

Consider the Hochschild-Serre spectral sequence of relative Lie algebra cohomology [12, I, §6] for the K_∞-stable ideal $\mathfrak{p}^- \subset \mathfrak{P}_h$:

$$E_2^{p,q} = H^p(\mathfrak{P}_h/\mathfrak{p}^-, K_\infty, H^q(\mathfrak{p}^-, W)) = H^p(\mathfrak{k}_{\infty,\mathbb{C}}, K_\infty, H^q(\mathfrak{p}^-, W)) \Rightarrow H^{p+q}(\mathfrak{P}_h, K_\infty, W).$$

Obviously, for any K_∞-module Y,

$$H^p(\mathfrak{k}_{\infty,\mathbb{C}}, K_\infty, Y) = \begin{cases} 0, & p > 0, \\ Y^{K_\infty}, & p = 0. \end{cases}$$

Thus the spectral sequence degenerates at E_2, and we have

$$H^p(\mathfrak{P}_h, K_\infty, W) \cong (H^p(\mathfrak{p}^-, W))^{K_\infty};$$

when $W = V \otimes V_\sigma$ this reduces to

$$H^p(\mathfrak{P}_h, K_\infty, V \otimes V_\sigma) \cong (H^p(\mathfrak{p}^-, V) \otimes V_\sigma)^{K_\infty}. \tag{4.1.4}$$

We say (π, V) is a *representation with $\bar{\partial}$-cohomology* if (π, V) has $\bar{\partial}$-cohomology with coefficients in σ for some (σ, V_σ).

(**4.1.5**) It follows from (4.1.2) that the inclusion $V_0 \subset V$ induces an isomorphism $H^*(\mathfrak{P}_h, K_\infty, V_0 \otimes V_\sigma) \xrightarrow{\sim} H^*(\mathfrak{P}_h, K_\infty, V \otimes V_\sigma)$ for any σ.

(**4.2**) In this paper V will either be a unitary $(\mathfrak{g}, K_\infty)$-module or a submodule of $C^\infty(G(\mathbf{A}))$. We begin with the latter case. Let (σ, V_σ) be a finite-dimensional representation of K_∞, and let E_σ^0 be the corresponding $G(\mathbf{A})$-homogeneous vector bundle on $X \times G(\mathbf{A}^f)$. Consider the bundle

$\Omega^{0,q}\otimes E^0_\sigma$ of $C^\infty(0, q)$ forms over $X\times G(\mathbf{A}^f)$ with values in E^0_σ. As in (3.3), there is a canonical isomorphism

(4.2.1)

$$\underline{\text{Lift}}\colon \Gamma^\infty(X\times G(\mathbf{A}^f), \Omega^{0,q}\otimes E^0_\sigma) \xrightarrow{\sim} \{f\in C^\infty(G(\mathbf{A}), V^q_\sigma) | f(gk) = \sigma^q(k)^{-1}f(g) \quad \forall g\in G(\mathbf{A}),\ k\in K_\infty\}.$$

We may naturally identify the right-hand side of (4.2.1) with

$$(C^q(\mathfrak{p}^-, C^\infty(G(\mathbf{A}))\otimes V_\sigma)^{K_\infty} = C^q(\mathfrak{P}_h, K_\infty, C^\infty(G(\mathbf{A}))\otimes V_\sigma).$$

The following lemma explains the terminology "$\bar\partial$-cohomology":

(4.2.2) *Lemma.* The following diagram is commutative:

$$\begin{array}{ccc} \Gamma^\infty(X\times G(\mathbf{A}^f), \Omega^{0,q}\otimes E^0_\sigma) & \xrightarrow{\bar\partial} & \Gamma^\infty(X\times G(\mathbf{A}^f), \Omega^{0,q+1}\otimes E^0_\sigma) \\ \big\downarrow\wr\ \underline{\text{Lift}} & & \big\downarrow\wr\ \underline{\text{Lift}} \\ C^q(\mathfrak{P}_h, K_\infty, C^\infty(G(\mathbf{A}))\otimes V_\sigma) & \xrightarrow{d} & C^{q+1}(\mathfrak{P}_h, K_\infty, C^\infty(G(\mathbf{A}))\otimes V_\sigma) \end{array}$$

Proof. This is well known. The diagram (without the contribution coming from the finite adeles) can be found on p. 109 of [39]; note that [39] uses the opposite complex structure to ours; their definition of E_σ is also dual to ours.

(**4.2.3**) The above lemma remains true if we replace $C^\infty(G(\mathbf{A}))$ by $C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})/K)$, and $X\times G(\mathbf{A}^f)$ by $G(\mathbb{Q})\backslash X\times G(\mathbf{A}^f)/K$ for any closed subgroup K of $G(\mathbf{A})$. In this form, the lemma will be applied to subspaces of $C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})/K)$ defined by various growth conditions. This will provide us with subcomplexes of $C^\cdot(\mathfrak{P}_h, K_\infty, C^\infty(G(\mathbf{A}))\otimes V_\sigma)$ which are not Lie algebra complexes.

(**4.3**) Let (π, V) be an irreducible admissible $(\mathfrak{g}, K_\infty)$-module. Let $Z(\mathfrak{g}_\mathbb{C})$ denote the center of the enveloping algebra $U(\mathfrak{g}_\mathbb{C})$. It is known that there exists a homomorphism $\chi_\pi\colon Z(\mathfrak{g}_\mathbb{C})\to\mathbb{C}$, the *infinitesimal character* of π, such that $\pi(z)=\chi_\pi(z)\ \ \forall z\in Z(\mathfrak{g}_\mathbb{C})$.

Any $\Lambda\in\mathfrak{h}^*$ naturally defines a homomorphism $e_\Lambda\colon S(\mathfrak{h})\to\mathbb{C}$. Let $\theta\colon Z(\mathfrak{g}_\mathbb{C})\xrightarrow{\sim} S(\mathfrak{h})^W$ be the Harish-Chandra isomorphism, where $W = W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ is the Weyl group [31, VIII, §5]. To $\Lambda\in\mathfrak{h}^*$ we associate the algebra homomorphism

(4.3.1) $$\chi_\Lambda = e_\Lambda\circ\theta\colon Z(\mathfrak{g}_\mathbb{C})\to\mathbb{C}.$$

Note that $\chi_\Lambda = \chi_{w\Lambda}\ \ \forall w\in W$. Then, for any irreducible admissible $(\mathfrak{g}, K_\infty)$-module π, it is known [31, Proposition 8.21] that $\chi_\pi = \chi_\Lambda$ for some $\Lambda\in\mathfrak{h}^*$, determined uniquely modulo the action of W.

Similarly, let $Z(\mathfrak{k}_{\infty,\mathbb{C}})$ denote the center of the enveloping algebra of $\mathfrak{k}_{\infty,\mathbb{C}}$, $W_C = W(\mathfrak{k}_{\infty,\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \subset W$. Let $\theta_{\mathfrak{k}}: Z(\mathfrak{k}_{\infty,\mathbb{C}}) \xrightarrow{\sim} S(\mathfrak{h})^{W_c}$ be the Harish-Chandra isomorphism for $\mathfrak{k}_{\infty,\mathbb{C}}$. As above, any $\Lambda \in \mathfrak{h}^*$ defines an algebra homomorphism $\chi'_\Lambda: Z(\mathfrak{k}_{\infty,\mathbb{C}}) \to \mathbb{C}$. If Λ is R_c^+-dominant and integral, let (σ, V_σ) be the irreducible finite-dimensional K_∞-module with highest weight Λ. We write $\sigma = \sigma_\Lambda$, $V_\sigma = V_\Lambda$. We let χ'_σ denote the infinitesimal character of σ; then $\chi'_\sigma = \chi'_{\Lambda+\rho_c}$.

The inclusion $W_c \subset W$ defines a *surjective* restriction map

$$\xi: \mathrm{Hom}_{\mathrm{alg}}(Z(\mathfrak{k}_{\infty,\mathbb{C}}), \mathbb{C}) \to \mathrm{Hom}_{\mathrm{alg}}(Z(\mathfrak{g}_{\mathbb{C}}), \mathbb{C})$$

such that $\xi(\chi'_\Lambda) = \chi_{\Lambda+\rho_n}$.

Let $\mathfrak{q} \subset \mathfrak{g}_{\mathbb{C}}$ be a parabolic subgroup with Levi decomposition $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{n}$. The Casselman-Osborne Lemma [15] provides necessary conditions for the cohomology groups $H^p(\mathfrak{n}, V)$ to contain given $\mathfrak{m}$-types. Applied to our situation, where $\mathfrak{q} = \mathfrak{P}_h$, $\mathfrak{n} = \mathfrak{p}^-$, $\mathfrak{m} = \mathfrak{k}_{\infty,\mathbb{C}}$, we obtain

(4.3.2) Proposition [15, *Theorem* 2.6]. *Let (π, V) be an irreducible admissible $(\mathfrak{g}, K_\infty)$-module. Let $(\sigma_\Lambda, V_\Lambda)$ be the finite-dimensional representation of K_∞ with highest weight Λ. Suppose (π, V) has $\bar{\partial}$-cohomology with coefficients in σ_Λ. Then $\chi_\pi = \xi(\chi'_{(\sigma_\Lambda)^*}) = \chi_{-\Lambda-\rho}$.*

(4.3.3) Corollary. *For a given finite-dimensional representation (σ, V_σ) of K, the number of irreducible admissible $(\mathfrak{g}, K_\infty)$-modules with $\bar{\partial}$-cohomology with coefficients in σ is finite.*

Proof. This follows from Harish-Chandra's well-known theorem that the number of irreducible admissible $(\mathfrak{g}, K_\infty)$-modules with given infinitesimal character is finite.

(4.3.4) Corollary. *Let $C_{\mathfrak{g}}$ denote the Casimir operator in $Z(\mathfrak{g}_{\mathbb{C}})$, and let $\langle\cdot,\cdot\rangle$ be the Killing form on $\mathfrak{g}^*$. Under the hypotheses of Proposition* (4.3.2), *we have*

$$\chi_\pi(C_{\mathfrak{g}}) = \langle\Lambda+\rho, \Lambda+\rho\rangle - \langle\rho, \rho\rangle.$$

(**4.4**) When (π, V) is *unitary*, Hodge theory provides a partial converse to Corollary (4.3.4). Thus, let (π, V) be a $(\mathfrak{g}, K_\infty)$-module, which is not yet assumed to be irreducible. We say (π, V) is *unitary* if there is a positive nondegenerate hermitian scalar product $(\cdot,\cdot)_\pi$ on V such that

$$(4.4.1) \qquad (Xv, w)_\pi + (v, Xw)_\pi = 0 \quad \forall X \in \mathfrak{g}^{\mathrm{der}}(\mathbb{R}),\ v, w \in V.$$

We extend $(\cdot,\cdot)_\pi$ linearly to $\mathfrak{g}(\mathbb{C})$; then (4.4.1) becomes

$$(4.4.2) \qquad (Xv, w)_\pi + (v, \bar{X}w)_\pi = 0 \quad \forall X \in \mathfrak{g}^{\mathrm{der}}(\mathbb{C}),\ v, w \in V,$$

where $X \mapsto \bar{X}$ is complex conjugation with respect to $\mathfrak{g}^{\mathrm{der}}(\mathbb{R})$. Recall that $Z_G(\mathbb{R}) \subset K_\infty$; thus any $(\mathfrak{g}, K_\infty)$-module is automatically $Z_G(\mathbb{R})$-semisimple.

Let $(\sigma_\Lambda, V_\Lambda)$ be the finite-dimensional representation of K_∞ with highest weight Λ. Choose K_∞-invariant hermitian inner products on V_Λ and on $\mathfrak{p}^-$; together with $(\cdot, \cdot)_\pi$, these define K_∞-invariant hermitian inner products on each of the terms of the complex $C^\cdot(\mathfrak{P}_h, K_\infty, V \otimes V_\Lambda)$. Let d_Λ^* denote the adjoint of d_Λ with respect to these inner products, and let $\Box_{\Lambda,\pi} = d_\Lambda d_\Lambda^* + d_\Lambda^* d_\Lambda$.

(4.4.3) Proposition (*Okamoto-Ozeki*). *On the complex*

$$C^\cdot(\mathfrak{P}_h, K_\infty, V \otimes V_\Lambda) = (\Lambda^\cdot(\mathfrak{p}^-)^* \otimes V \otimes V_\Lambda)^{K_\infty},$$

we have the formula

$$\Box_{\Lambda,\pi} = \tfrac{1}{2}(c_\Lambda - 1 \otimes \pi(C_\mathfrak{g}) \otimes 1),$$

where $c_\Lambda = \langle \Lambda + \rho, \Lambda + \rho \rangle - \langle \rho, \rho \rangle$.

Proof. This proposition is stated as Theorem 4.1 of [39] in the case $V = L_2(G_0)^\infty$, the space of square-integrable C^∞ functions on G_0. However, the only property of this representation used in the proof is (4.4.2).

Now we take π to be irreducible, with infinitesimal character χ_π. Then the spaces $C^\cdot(\mathfrak{P}_h, K_\infty, V \otimes V_\Lambda)$ are all *finite-dimensional.* Thus the proof of [12, II, Proposition 3.1] goes over word for word in our case, and we obtain:

(4.5) Proposition. *Let (π, V) be an irreducible unitary $(\mathfrak{g}, K_\infty)$-module, and let $(\sigma_\Lambda, V_\Lambda)$ be the irreducible representation of K_∞ with highest weight Λ. Let $c_\Lambda = \langle \Lambda + \rho, \Lambda + \rho \rangle - \langle \rho, \rho \rangle$.*

(a) *If $\chi_\pi(C_\mathfrak{g}) \neq c_\Lambda$, then (π, V) has no $\bar{\partial}$-cohomology with coefficients in σ_Λ.*

(b) *If $\chi_\pi(C_\mathfrak{g}) = c_\Lambda$, then all cochains in the complex $C^\cdot(\mathfrak{P}_h, K_\infty, V \otimes V_\Lambda)$ are closed,*

$$H^q(\mathfrak{P}_h, K_\infty, V \otimes V_\Lambda) = C^q(\mathfrak{P}_h, K_\infty, V \otimes V_\Lambda) \cong \mathrm{Hom}_{K_\infty}(\Lambda^q(\mathfrak{p}^-) \otimes V_\Lambda^*, V), \quad q = 0, \cdots, n,$$

and every class in $H^q(\mathfrak{P}_h, K_\infty, V \otimes V_\Lambda)$ has a unique $\Box_{\Lambda,\pi}$-harmonic representative.

(4.6) We now describe the $\bar{\partial}$-cohomology of $(\mathfrak{g}, K_\infty)$-modules associated to representations in the discrete series and limits of discrete series, following [5]. Let $\mathcal{F} \subset \mathfrak{h}_\mathbb{C}^*$ denote the set of differentials of *algebraic* characters of the torus $H \subset K_\infty$ (notation (2.0)). Then every $\lambda \in \mathcal{F}$ satisfies

$2\langle\lambda, \alpha\rangle/\langle\alpha, \alpha\rangle \in \mathbb{Z}$, $\forall\alpha \in R$. Let $\mathscr{F}+\rho \subset \mathfrak{h}_{\mathbb{C}}^*$ be the set $\{\Lambda+\rho | \Lambda \in \mathscr{F}\}$. Choose a system of positive roots $\psi \subset R$, $\psi \supset R_c^+$, such that λ is dominant relative to ψ, and suppose $\lambda \in \mathscr{F}+\rho$ satisfies

(4.6.1) $\quad \langle\lambda, \alpha\rangle > 0 \quad \forall\alpha \in R_c^+$ such that α is *simple* with respect to ψ.

Then we may define the *limit of discrete series* $\pi(\lambda, \psi)$ as in [31, XII, §7]. If λ is nonsingular for R, then ψ is uniquely determined, and $\pi(\lambda, \psi)$ is the discrete series π_λ with Harish-Chandra parameter λ. We let $V(\lambda, \psi)$ (resp. V_λ) be the $(\mathfrak{g}, K_\infty)$-module associated to $\pi(\lambda, \psi)$ (resp. to π_λ). Let $R_n^+(\psi) = R_n \cap \psi$, $Q_{\lambda,\psi} = R_n^+ \cap \psi$, and $q_{\lambda,\psi} = |Q_{\lambda,\psi}|$. If λ is nonsingular, we write $Q_\lambda = Q_{\lambda,\psi}$, $q_\lambda = q_{\lambda,\psi}$.

The proof of the following theorem will appear in [5].

(4.6.2) Theorem (*Blasius, Harris, and Ramakrishnan* [5]. *Let ψ and λ be as above, and suppose λ is R_c-regular; i.e., $\langle\lambda, \alpha\rangle \neq 0$ $\forall\alpha \in R_c$. Let (σ_τ, V_τ) be the finite-dimensional irreducible representation of K_∞ with highest weight τ. Then*:

(i) *the character $\Lambda = \lambda - \rho$ is R_c^+-dominant and integral,*

(ii) $H^q(\mathfrak{P}_h, K_\infty, (\pi(\lambda, \psi))^* \otimes V_\tau) = 0$ *unless* $q = q_{\lambda,\psi}$ *and* $\tau = \Lambda$,

(iii) $\dim H^{q_{\lambda,\psi}}(\mathfrak{P}_h, K_\infty, (\pi(\lambda, \psi))^* \otimes V_\Lambda) = 1$.

The $\mathfrak{n}$-cohomology of discrete series modules has been computed by Schmid [43]; Schmid's result has been extended to the limits of discrete series considered above by Williams [55]. Theorem (4.6.2) is equivalent to these results in the Hermitian symmetric case.

The following partial converse to Theorem (4.6.2) seems to be well known.

(4.7) Lemma. *There is a constant $b \geq 0$ with the following property. Assume $|\langle\Lambda+\rho, \alpha\rangle| > b$ for all $\alpha \in R$. Let (π, V) be an irreducible unitary $(\mathfrak{g}, K_\infty)$-module. Assume (π, V) has $\bar\partial$-cohomology in degree q with coefficients in σ_Λ. Then*:

(i) *π^* is the discrete series representation $\pi_{\Lambda+\rho}$ with (regular) Harish-Chandra parameter $\Lambda+\rho$,*

(ii) *$q = q_{\Lambda+\rho,\psi}$, where ψ is the unique system of positive roots with respect to which $\Lambda+\rho$ is dominant, and*

(iii) $\dim H^{q_{\Lambda+\rho,\psi}}(\mathfrak{P}_h, K_\infty, \pi^* \otimes V_\Lambda) = 1$.

Proof. We prove (i); assertions (ii) and (iii) then follow from Theorem (4.6.2). Using Proposition (4.5) we obtain that

(4.7.1) $$\chi_\pi(C_\mathfrak{g}) = c_\Lambda;$$

(4.7.2) $$\mathrm{Hom}_{K_\infty}((V_\sigma^q)^*, V) \neq 0.$$

It suffices to show that V is in the discrete series, since conditions (4.7.1) and (4.7.2) uniquely characterize $(\pi_{\Lambda+\rho})^*$ among discrete series representations for b sufficiently large [37, §7]. Already for $b = 0$, conditions (4.7.1) and (4.7.2) imply [56, Lemma 2.2] that V is a derived functor module in the sense of [53], hence a unitary representation with nonzero (twisted) $(\mathfrak{g}, K_\infty)$-cohomology [52]. Assume this to be the case. Then Theorems II.6.12 and II.7.3 of [12] provide conditions on Λ which imply that $\mathscr{V}$ is discrete series (the notation Λ in [12] corresponds to our $\Lambda + \rho$). Let W^1 be the subgroup of $w \in W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ such that $w^{-1}(R_c^+) \subset R^+$. There is a finite collection $\{\beta\}$ of weights of $(\mathfrak{k}_{\infty,\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ such that if

(4.7.3) $\quad t(\Lambda+\rho) - \beta$ is R_c^+-dominant integral for all β, all $t \in W^q$,

then the conditions of [12] are satisfied. Obviously b can be chosen so that (4.7.3) is satisfied whenever $|\langle\Lambda, \alpha\rangle| > b$ for all $\alpha \in R$.

This lemma can also be derived from Theorem 1.9 of the article [56] of Williams to which we alluded above.

(4.8) Remark. It is natural to ask for a classification of unitary representations with nontrivial $\bar{\partial}$-cohomology. The proof of Theorem (4.6.2) reduces this to the problem of classifying unitary representations with nontrivial $\mathfrak{n}$-cohomology; this latter problem can be posed for an arbitrary real reductive group G with $\operatorname{rank} G = \operatorname{rank} K$. This problem is apparently not significantly easier than the problem of classifying arbitrary unitary representations.

5. Representing cohomology by automorphic forms

(5.1) Notation. For any quasicharacter $\chi\colon Z_G(\mathbf{A})/Z_G(\mathbb{Q}) \to \mathbb{C}^\times$, let

$$C_\chi = C_\chi(G) = \{f \in C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})) | f(gz) = \chi(z)f(g)\ \forall z \in Z_G(\mathbf{A}),\ g \in G(\mathbf{A})\}.$$

For each such χ, there is a unique quasicharacter $\xi_\chi\colon G(\mathbf{A}) \to \mathbb{R}^\times$ such that $\xi_\chi(z) = |\chi(z)|$, $\forall z \in Z_G(\mathbf{A})$. We let $f \otimes \xi_\chi^{-1}(g) = f(g)\xi_\chi(g)^{-1}$, and define

$$C_{(2),\chi} = C_{(2),\chi}(G) = \left\{ f \in C_\chi \,\middle|\, \int_{Z_G(\mathbf{A})G(\mathbb{Q})\backslash G(\mathbf{A})} |f \otimes \xi_\chi^{-1}(g)|^2\, dg < \infty \right\}.$$

We say $f \in C_\chi$ is *square-integrable* if $f \in C_{(2),\chi}$. More generally, suppose $f \in C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))$ is a finite sum $f = \sum f_\chi$, $f_\chi \in C_\chi$. We say f is *square-integrable* if each f_χ is, and we let $C_{(2)} = C_{(2)}(G) = \bigoplus_\chi C_{(2),\chi}$.

The space $C_{(2)}$ is endowed with a Hermitian inner product by decreeing that $C_{(2),\chi}$ and $C_{(2),\chi'}$ are orthogonal if $\chi \neq \chi'$, and by setting

$$(5.1.1)\quad (f, f') = \int_{Z_G(\mathbf{A})G(\mathbb{Q})\backslash G(\mathbf{A})} f(g)\overline{f'}(g)\xi_\chi(g)^{-2}\,dg \quad \forall f, f' \in C_{(2),\chi}.$$

We denote by C_{si} (resp. C_{rd}) the space of all slowly increasing (resp. rapidly decreasing) C^∞ functions on $G(\mathbb{Q})\backslash G(\mathbf{A})$ in the sense of (3.3); then $C_{\mathrm{rd}} \subset C_{(2)} \subset C_{\mathrm{si}}$. Note that C_* is *not* a $\mathfrak{g}$-module for $* = \mathrm{si}$, (2), or rd.

For any irreducible representation (σ, V_σ) of K_∞ and $0 \le q \le n$, we define C_σ^q as in (3.3). When σ is the trivial representation σ_0, we write V^q and $C^q = C^q(G)$ instead of $V_{\sigma_0}^q$ and $C_{\sigma_0}^q$. Then $C^\cdot(G)$ is a complex under $\bar{\partial}$, the image under the homomorphism Lift of (3.3) of the Dolbeault complex of the structure sheaf of $M(G, X)(\mathbb{C})$. For each q, there is a natural map

$$L^q : C^q \otimes (V^q)^* \hookrightarrow C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})).$$

For $* = \mathrm{rd}$, (2), or si, we let

$$(5.1.2)\qquad C_*^q = \{f \in C^q \mid L^q(f \otimes (V^q)^*) \text{ and } L^{q+1}(\bar{\partial} f \otimes (V^{q+1})^*) \subset C_*\}.$$

Then $C_*^\cdot$ is a complex under $\bar{\partial}$, for $* = \mathrm{rd}$, (2), or si.

More generally, for any irreducible representation (σ, V_σ) of K_∞, we let $C_{*,\sigma}^\cdot = (C_*^\cdot \otimes V_\sigma)^{K_\infty}$ for $* = \mathrm{si}$, (2), or rd. This is consistent with the notation introduced in (3.3). For $0 \le q \le n$, $* = \mathrm{si}$, (2), or rd, we let $Z_{*,\sigma}^q$ (resp. $B_{*,\sigma}^q$) denote the subspace of $\bar{\partial}$-closed forms (resp. $\bar{\partial}$-exact forms) in $C_{*,\sigma}^q$. For every level subgroup $K \subset G(\mathbf{A}^f)$, we define ${}_K C_{*,\sigma}^q$ (resp. ${}_K Z_{*,\sigma}^q$, resp. ${}_K B_{*,\sigma}^q$) to be the space of K-fixed vectors in $C_{*,\sigma}^q$ (resp. $Z_{*,\sigma}^q$, resp. $B_{*,\sigma}^q$).

Let $L_{(2),\sigma}^q$ (resp. $L_{(2)}^q$, resp. $L_{(2)}$) be the completion of $C_{(2),\sigma}^q$ (resp. $C_{(2)}^q$, resp. $C_{(2)}$) with respect to the inner product (5.1.1) on $C_{(2)}$ and the given inner products on V^q and V_σ^q. For every level subgroup $K \subset G(\mathbf{A}^f)$, define ${}_K L_{(2),\sigma}^q$ as above.

Langlands' theory of Eisenstein series [33] provides a decomposition

$$L_{(2)} \cong L_{(2),\mathrm{disc}} \oplus L_{(2),\mathrm{cont}},$$

where disc and cont refer to the discrete and continuous spectra, respectively. The space $L_{(2),\mathrm{disc}}$ is a Hilbert direct sum of a countable family of irreducible unitary representation of $G(\mathbf{A})$. Similarly, given an irreducible representation σ of K_∞, a level subgroup $K \subset G(\mathbf{A}^f)$, and $0 \le q \le n$, we obtain a decomposition

$$(5.1.3) \qquad {}_K L^q_{(2),\sigma} \cong {}_K L^q_{(2),\sigma,\text{disc}} \oplus {}_K L^q_{(2),\sigma,\text{cont}}.$$

We have the further decomposition

$$(5.1.4) \qquad L^q_{(2),\sigma,\text{disc}} \cong L^q_{(2),\sigma,\text{cusp}} \oplus L^q_{(2),\sigma,\text{res}},$$

where cusp and res refer to the cuspidal and residual spectra, respectively.

(**5.2**) Let $\mathscr{A}(G)$ (resp. $\mathscr{A}_{(2)}(G)$, resp. $\mathscr{A}_0(G)$) denote the space of all automorphic forms (resp. square-integrable automorphic forms, resp. cusp forms) on $G(\mathbb{Q})\backslash G(\mathbf{A})$ in the sense of [11]. Thus $\mathscr{A}(G)$ is the $(\mathfrak{g}, K_\infty)$-submodule of K_∞-finite and $Z(\mathfrak{g}_\mathbb{C})$-finite vectors in C_{si}. Since $L_{(2),\text{cont}}$ contains no $Z(\mathfrak{g}_\mathbb{C})$-finite functions, it follows that

$$(5.2.1) \qquad \mathscr{A}_{(2)}(G) \subset L_{(2),\text{disc}}.$$

By Lemma (4.2.2), we have the following diagram of complexes:

(5.2.2)

$$\begin{array}{ccccccc}
C^\bullet(\mathfrak{P}_h, K_\infty, \mathscr{A}_0(G)\otimes V_\sigma) & \hookrightarrow & C^\bullet(\mathfrak{P}_h, K_\infty, \mathscr{A}_{(2)}(G)\otimes V_\sigma) & \hookrightarrow & C^\bullet(\mathfrak{P}_h, K_\infty, \mathscr{A}(G)\otimes V_\sigma) & & \\
\searrow & & \swarrow & \swarrow & \downarrow & & \\
C^\bullet_{\text{rd},\sigma} & \hookrightarrow & C^\bullet_{(2),\sigma} & \hookrightarrow & C^\bullet_{\text{si},\sigma} & \hookrightarrow & C^\bullet(\mathfrak{P}_h, K_\infty, C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))\otimes V_\sigma)
\end{array}$$

As in (3.5), we let $\mathscr{H}^q_{\text{cusp},\sigma}$ (resp. $\mathscr{H}^q_{(2),\sigma}$) denote the space of harmonic cusp forms (resp. harmonic square-integrable forms) in $C^q_{\text{rd},\sigma}$ (resp. $C^q_{(2),\sigma}$).

(5.2.3) Lemma. *There are natural isomorphisms*

$$\begin{aligned}
\mathscr{H}^*_{\text{cusp},\sigma} &\xrightarrow{\sim} H^*(\mathfrak{P}_h, K_\infty, \mathscr{A}_0(G)\otimes V_\sigma) \\
&= \bigoplus_{(\pi,V)} H^*(\mathfrak{P}_h, K_\infty, V\otimes V_\sigma)\otimes \operatorname{Hom}_{(\mathfrak{g},K_\infty)}(V, \mathscr{A}_{(0)}(G)), \\
\mathscr{H}^*_{(2),\sigma} &\xrightarrow{\sim} H^*(\mathfrak{P}_h, K_\infty, \mathscr{A}_{(2)}(G)\otimes V_\sigma) \\
&= \bigoplus_{(\pi,V)} H^*(\mathfrak{P}_h, K_\infty, V\otimes V_\sigma)\otimes \operatorname{Hom}_{(\mathfrak{g},K_\infty)}(V, \mathscr{A}_{(2)}(G)),
\end{aligned}$$

where (π, V) *runs through the set of unitary* $(\mathfrak{g}, K_\infty)$*-modules.*

Proof. Both $\mathscr{A}_0(G)$ and $\mathscr{A}_{(2)}(G)$ are countable direct sums of irreducible unitary $(\mathfrak{g}, K_\infty)$-modules. The lemma thus follows from Proposition (4.5) and Lemma (4.2.2).

By Lemmas (4.2.2) and (5.2.3), and Corollary (3.4), we may thus translate (5.2.2) into the following commutative diagram of cohomology groups:

(5.2.4)
$$\begin{array}{ccccccc} \mathscr{H}^*_{\mathrm{cusp},\sigma} & \to & \mathscr{H}^*_{(2),\sigma} & \to & H^*(\mathfrak{P}_h, K_\infty; \mathscr{A}(G)\otimes V_\sigma) & & \\ \downarrow & & \downarrow & & \downarrow & \searrow & \\ \tilde H^*([\check E_\sigma]^{\mathrm{sub}}) & \to & H^*_{(2),\sigma} & \to & \tilde H^*([\check E_\sigma]^{\mathrm{can}}) & \to & H^*(M(G,X),[\check E_\sigma]). \end{array}$$

Here $H^*_{(2),\sigma} = Z^*_{(2),\sigma}/B^*_{(2),\sigma}$. All the arrows in (5.2.4) are homomorphisms of $G(\mathbf{A}^f)$-modules.

(**5.2.5**) We now apply some results of Borel and Garland. Let $C_{\mathfrak g}$ be the Casimir operator of $\mathfrak g$, as in §4. We may write

$$L^{\cdot}_{(2),\sigma,\mathrm{disc}} = \hat{\bigoplus}_{\xi} L^{\cdot}_{(2),\sigma,\mathrm{disc}}(\xi) \tag{5.2.5.1}$$

as a countable Hilbert direct sum of complexes, where $L^{\cdot}_{(2),\sigma,\mathrm{disc}}(\xi)$ is the subspace of $L^{\cdot}_{(2),\sigma,\mathrm{disc}}$ of eigenfunctions for $C_{\mathfrak g}$ with eigenvalue ξ. As in (5.1.4), we may write

$$L^q_{(2),\sigma,\mathrm{disc}}(\xi) \cong L^q_{(2),\sigma,\mathrm{disc}}(\xi) \oplus L^q_{(2),\sigma,\mathrm{res}}(\xi) \quad \forall \xi \in \mathbb{R}.$$

Borel and Garland have proved that each ${}_K L^{\cdot}_{(2),\sigma,\mathrm{disc}}(\xi)$ is finite dimensional [10, Theorem 4.6], and consists of $Z(\mathfrak g_{\mathbb C})$-finite functions [10, Proposition 4.3]. For each ξ, let $\square_\xi$ be the Laplacian on the complex ${}_K L^{\cdot}_{(2),\sigma,\mathrm{disc}}(\xi)$. Let $c_\Lambda = \langle \Lambda+\rho, \Lambda+\rho\rangle - \langle\rho,\rho\rangle$, as in (4.4). For $\xi \neq c_\Lambda$, $\square_\xi$ is invertible, hence (cf. [10, Lemma 5.2])

(5.2.5.2)
$$\begin{aligned} H^*({}_K L^{\cdot}_{(2),\sigma,\mathrm{disc}}) &= H^*({}_K L^{\cdot}_{(2),\sigma,\mathrm{disc}}(c_\Lambda)) \\ &= H^*({}_K L^{\cdot}_{(2),\sigma,\mathrm{cusp}}(c_\Lambda)) \oplus H^*({}_K L^{\cdot}_{(2),\sigma,\mathrm{res}}(c_\Lambda)). \end{aligned}$$

It follows that $H^*({}_K L^{\cdot}_{(2),\sigma,\mathrm{disc}})$ and $H^*({}_K L^{\cdot}_{(2),\sigma,\mathrm{cusp}})$ are represented by $Z(\mathfrak g_{\mathbb C})$-finite forms. Taking the direct limit over K, from (5.2.1) thus follows

(5.2.6) Lemma. *We have* $H^*(L^{\cdot}_{(2),\sigma,\mathrm{disc}}) = \mathscr{H}^*_{(2),\sigma}$ *and* $H^*(L^{\cdot}_{(2),\sigma,\mathrm{cusp}}) = \mathscr{H}^*_{\mathrm{cusp},\sigma}$.

We are now in a position to state our main theorem.

(5.3) Theorem. *Let* (σ, V_σ) *be a representation of* K_∞, *and let* $[\mathscr{V}] = [\check E_\sigma]$ *be the corresponding fully decomposed automorphic vector bundle. Let* $\bar H^*([\mathscr{V}])$ *denote the image of* $\tilde H^*([\mathscr{V}]^{\mathrm{sub}})$ *in* $\tilde H^*([\mathscr{V}]^{\mathrm{can}})$. *Then* $\bar H^*([\mathscr{V}])$

is contained in the image of $H^*(L^{\cdot}_{(2),\sigma,\mathrm{disc}}) = \mathscr{H}^*_{(2),\sigma}$ *in* $\tilde{H}^*([\mathscr{V}]^{\mathrm{can}})$, *under the morphism obtained from the diagram* (5.2.4).

The proof of Theorem (5.3) will be given in §6. In the rest of this section, we derive a few of its more important consequences. We begin by paraphrasing a theorem of Wallach which will be used repeatedly.

(5.3.1) Theorem (*Wallach* [54]). *Let* $\mathscr{V}$ *be a tempered* $(\mathfrak{g}, K_\infty)$-*module. Let* $T\colon \mathscr{V} \to \mathscr{A}(G) \cap L_2(G(\mathbb{Q})\backslash G(\mathbf{A})) = \mathscr{A}_{(2)}(G)$ *be a* $(\mathfrak{g}, K_\infty)$-*module homomorphism. Then* $T(\mathscr{V}) \subset \mathscr{A}_0(G)$.

Our main theorem has the following immediate corollary.

(5.3.2) Corollary. *Suppose the representation* (σ, V_σ) *and the integer* q *satisfy the following conditions*:

(a) *For every irreducible constituent* σ' *of* σ, *there is at most one irreducible unitary* $(\mathfrak{g}, K_\infty)$-*module* $(\pi(\sigma'), V(\sigma'))$ *such that* $(\pi(\sigma'), V(\sigma'))$ *has* $\bar\partial$-*cohomology in degree* q *with coefficients in* σ'.

(b) *Each of the* $(\pi(\sigma'), V(\sigma'))$ *is a tempered* $(\mathfrak{g}, K_\infty)$-*module.*

Then the homomorphism $\mathscr{H}^q_{\mathrm{cusp},\sigma} \to \bar{H}^q([\mathscr{V}])$ *of Proposition* (3.6) *is an isomorphism.*

Proof. It follows from Proposition (3.6) and Theorem (5.3) that we have a diagram of inclusions

$$\mathscr{H}_{\mathrm{cusp},\sigma} \hookrightarrow \bar{H}^q([\mathscr{V}]) \hookrightarrow \mathrm{Im}(\mathscr{H}^q_{(2),\sigma} \to \tilde{H}^q([\mathscr{V}]^{\mathrm{can}})).$$

Now Lemma (5.2.3) provides an isomorphism

$$\mathscr{H}^q_{(2),\sigma} \xrightarrow{\sim} H^q(\mathfrak{P}_h, K_\infty, \mathscr{A}_{(2)}(G) \otimes V_\sigma).$$

But our hypotheses, together with Theorem (5.3.1), imply that the natural map

$$H^q(\mathfrak{P}_h, K_\infty, \mathscr{A}_0(G) \otimes V_\sigma) \to H^q(\mathfrak{P}_h, K_\infty, \mathscr{A}_{(2)}(G) \otimes V_\sigma)$$

is an isomorphism. Hence the corollary follows immediately.

Lemma (4.7) provides an important special case of Corollary (5.3.2):

(5.3.3) Corollary. *There is a constant* $b \geq 0$ *with the following property*: *Assume every irreducible constituent of* σ *has highest weight* Λ, *with* $|\langle \Lambda + \rho, \alpha \rangle| > b$ *for every* $\alpha \in R$. *Then the homomorphism* $\mathscr{H}^*_{\mathrm{cusp},\sigma} \to \bar{H}^*([\mathscr{V}])$ *of Proposition* (3.6) *is an isomorphism. In particular, if we write* $\sigma = \bigoplus \sigma_\Lambda$ *and* $[\mathscr{V}] = \bigoplus [\mathscr{V}_\Lambda]$, *then*

$$\bar{H}^q([\mathscr{V}_\Lambda]) = 0, \qquad q \neq q_{\Lambda+\rho},$$

and there is a natural isomorphism of $G(\mathbf{A}^f)$-*modules*:

$$\bar{H}^q([\mathscr{V}]) \cong \bigoplus_{q_{\Lambda+\rho}=q} \mathrm{Hom}_{(\mathfrak{g}, K_\infty)}((\pi_{\Lambda+\rho})^*, \mathscr{A}_0(G)), \tag{5.3.4}$$

where Λ *runs through the set of highest weights in* σ.

Proof. It follows from Lemma (4.7) that, for every q, $\sigma = \bigoplus \sigma_\Lambda$ satisfies condition (a) of Corollary (5.3.2), where $\pi(\sigma_\Lambda)$ is the discrete series representation $(\pi_{\Lambda+\rho})^*$. Since discrete series are tempered, Corollary (5.3.2) applies. Following Lemma (5.2.3), we may thus write

$$(5.3.5)\qquad \begin{aligned} \bar H^q([\mathscr{V}]) &\cong \mathscr{H}^q_{\mathrm{cusp},\sigma} \\ &\cong \bigoplus_\Lambda H^q(\mathfrak{P}_h, K_\infty, (\pi_{\Lambda+\rho})^* \otimes V_\Lambda) \\ &\qquad \otimes \mathrm{Hom}_{(\mathfrak{g}, K_\infty)}((\pi_{\Lambda+\rho})^*, \mathscr{A}_0(G)) \end{aligned}$$

as $G(\mathbf{A}^f)$-modules, where Λ runs through the set of highest weights in σ. The corollary now follows from Lemma (4.7).

(5.4) Theorem. *Let (π, V) be an irreducible unitary $(\mathfrak{g}, K_\infty)$-module with the following properties:*

(a) *There exist a finite-dimensional irreducible representation (σ, V_σ) of K_∞ and an integer q such that (π, V) has $\bar\partial$-cohomology in degree q with coefficients in σ.*

(b) *No other unitary $(\mathfrak{g}, K_\infty)$-module has $\bar\partial$-cohomology in degree q with coefficients in σ.*

(c) *The module (π, V) is tempered.*

(d) $\dim H^q(\mathfrak{P}_h, K_\infty, V \otimes V_\sigma) = 1$.

Let $E(\sigma)$ be the field of definition of the G-homogeneous vector bundle $\check E_\sigma$ over $\check M$, associated to σ (notation (2.1)). Then the space $\mathrm{Hom}_{(\mathfrak{g}, K_\infty)}(V, \mathscr{A}_0(G))$ *has a $G(\mathbf{A}^f)$-invariant $E(\sigma)$-rational structure.*

Remark. Hypothesis (d) in Theorem (5.4) is included for convenience. The most interesting examples for our present purposes—discrete series and limits of discrete series—do satisfy hypothesis (d) (Theorem (4.6.2)).

This theorem has interesting applications to eigenvalues of Hecke operators [5].

Proof. Under the hypotheses of the theorem, the representation σ satisfies the hypotheses of Corollary (5.3.2). As in (5.3.5), we thus have

$$\begin{aligned} \bar H^q([\mathscr{V}]) &\cong H^q(\mathfrak{P}_h, K_\infty, V \otimes V_\sigma) \otimes \mathrm{Hom}_{(\mathfrak{g}, K_\infty)}(V, \mathscr{A}_0(G)) \\ &\cong \mathrm{Hom}_{(\mathfrak{g}, K_\infty)}(V, \mathscr{A}_0(G)). \end{aligned}$$

It seems reasonable to make the following conjecture:

(5.4.1) Conjecture. Let $\mathscr{V}$ be a G-homogeneous, fully decomposed vector bundle on $\check M(G, X)$, rational over the field $E(\mathscr{V})$. The image of the homomorphism $\mathrm{cl}\colon \mathscr{H}^q_{\mathrm{cusp}} \to \bar H^q([\mathscr{V}])$ of Proposition (3.6) is rational over $E(\mathscr{V})$ for every integer q.

In addition to the cases covered by Corollary (5.3.2), the conjecture is trivially satisfied in the cases $q = 0$ and $q = n$:

(5.4.2) Proposition. *For $q = 0$ or $q = n$, the homomorphism* $\mathrm{cl}\colon \mathscr{H}^q_{\mathrm{cusp}} \to \bar{H}^q([\mathscr{V}])$ *is an isomorphism for any G-homogeneous, fully decomposed vector bundle $\mathscr{V}$ on $\check{M}(G, X)$.*

Proof. It suffices to verify that the homomorphism ${}_K\mathscr{H}^q_{\mathrm{cusp}} \to \bar{H}^q_K([\mathscr{V}])$ is an isomorphism for every level subgroup K for $q = 0$ or $q = n$. Fix a level subgroup K, and choose an SNC toroidal compactification $\bar{M}$ of $M = {}_K M(G, X)$; let M^* be the Satake-Baily-Borel minimal compactification. Let $r\colon \bar{M} \to M^*$ and $j\colon M \hookrightarrow M^*$ be the natural maps.

Suppose first that $q = 0$. Then $\bar{H}^0_K([\mathscr{V}]) \cong H^0_K([\mathscr{V}]^{\mathrm{sub}}) = H^0(\bar{M}, [\mathscr{V}]^{\mathrm{sub}})$, which is by definition the space of holomorphic global sections of $[\mathscr{V}]^{\mathrm{can}}$ over $\bar{M}$ which vanish along Z. Now Mumford has verified [36, Proposition 3.3] that the image of the restriction map $H^0(\bar{M}, [\mathscr{V}]^{\mathrm{can}}) \to H^0(M, [\mathscr{V}]) \cong H^0(M^*, j_*[\mathscr{V}])$ is the subspace consisting of sections regular at infinity; the latter condition is vacuous unless G contains a factor isogenous to $\mathrm{SL}(2, \mathbb{Q})$. Thus, under the natural map $H^0(\bar{M}, [\mathscr{V}]^{\mathrm{can}}) \to H^0(M^*, j_*[\mathscr{V}])$, the subspace $H^0(\bar{M}, [\mathscr{V}]^{\mathrm{sub}})$ is taken to the space of sections of $H^0(M^*, j_*[\mathscr{V}])$ which vanish on $M^* - M$. But this is exactly the space of cusp forms (cf. [3, §10]).

Now suppose $q = n$. Define $[\mathscr{V}]' = \mathbb{K} \otimes [\mathscr{V}]^*$ as in Corollary (2.3). Say $[\mathscr{V}]$ (resp. $[\mathscr{V}]'$) is the automorphic vector bundle associated to the representation σ (resp. τ) of K_∞. Up to twisting by a character, complex conjugation defines an antilinear isomorphism ${}_K\mathscr{H}^0_{\mathrm{cusp},\tau} \xrightarrow{\sim} {}_K\mathscr{H}^n_{\mathrm{cusp},\sigma}$; in particular the two spaces have the same dimension. On the other hand, by Serre duality (2.3) $\dim H^n(\bar{M}, [\mathscr{V}]^{\mathrm{can}}) = \dim H^0(\bar{M}, [\mathscr{V}]^{\prime,\mathrm{sub}})$. We have already verified that $\dim H^0(\bar{M}, [\mathscr{V}]^{\prime,\mathrm{sub}}) = \dim {}_K\mathscr{H}^0_{\mathrm{cusp},\tau}$; thus $\dim {}_K\mathscr{H}^n_{\mathrm{cusp},\sigma} = \dim H^n(\bar{M}, [\mathscr{V}]^{\mathrm{can}})$. Since the map ${}_K\mathscr{H}^n_{\mathrm{cusp},\sigma} \to \bar{H}^n_K([\mathscr{V}]) \subset H^n(\bar{M}, [\mathscr{V}]^{\mathrm{can}})$ is injective, by Proposition (3.6), cl must be an isomorphism.

6. Proof of Theorem (5.3)

We retain the notation of §5. For any level subgroup $K \subset G(\mathbf{A}^f)$, we let $\mathscr{H}(G(\mathbf{A}^f), K)$ denote the Hecke algebra of K-bi-invariant functions on $G(\mathbf{A}^f)$, as in the proof of Theorem (5.4). The algebra $\mathscr{H}(G(\mathbf{A}^f), K)$ acts naturally on all the spaces ${}_K L^q_{(2),\sigma}$, ${}_K C^q_{*,\sigma}$, etc., which were introduced in (5.1).

Theorem (5.3) is actually a simple consequence of the following lemma:

(6.1) Lemma. *Every $\mathscr{H}(G(\mathbf{A}^f), K)$-finite vector in ${}_K L^{\cdot}_{(2),\sigma}$ belongs to ${}_K L^{\cdot}_{(2),\sigma,\mathrm{disc}}$*

We admit the truth of Lemma (6.1) for the moment, and derive Theorem (5.3). It suffices to prove that, for every level subgroup K and $0 \leq q \leq n$, the image $\bar{H}^q_K([\mathscr{V}])$ of $\tilde{H}^q_K([\mathscr{V}]^{\mathrm{sub}})$ in $\tilde{H}^q_K([\mathscr{V}]^{\mathrm{can}})$ is contained in the image of $H^q({}_K L^{\cdot}_{(2),\sigma,\mathrm{disc}}) = \mathscr{H}^q_{(2),\sigma} \cap {}_K L^q_{(2),\sigma,\mathrm{disc}}$ in $\tilde{H}^q_K([\mathscr{V}]^{\mathrm{can}})$. Fix a level subgroup K and an integer q. The diagram (5.2.4) shows that $\bar{H}^q_K([\mathscr{V}])$ is at least contained in the image of the natural homomorphism

$$ {}_K Z^q_{(2),\sigma} \to \tilde{H}^q_K([\mathscr{V}]^{\mathrm{can}}). $$

We denote this homomorphism λ.

(6.2) Lemma. *The homomorphism λ is continuous in the L_2 topology* (5.1.1). *It thus extends to a homomorphism*

$$ \bar{\lambda}\colon\ {}_K \bar{Z}^q_{(2),\sigma} \to \tilde{H}^q_K([\mathscr{V}]^{\mathrm{can}}) $$

with closed kernel, where ${}_K \bar{Z}^q_{(2),\sigma}$ is the closure of ${}_K Z^q_{(2),\sigma}$ in ${}_K L^q_{(2),\sigma}$.

Proof. It suffices to show that for a basis L_α of the dual space of $\tilde{H}^q_K([\mathscr{V}]^{\mathrm{can}})$, the composite homomorphisms $L_\alpha \circ \lambda$ are continuous in the L_2 topology. Now the dual space is just $\tilde{H}^{n-q}_K([\mathscr{V}]^{\prime,\mathrm{sub}})$, by Corollary (2.3). Let τ be the representation of K_∞ corresponding to $[\mathscr{V}]'$. It follows from Proposition (3.8) that there is a constant C and, for each α, a function $f_\alpha \in {}_K C^{n-q}_{\mathrm{rd},\tau}$ such that

$$ L_\alpha(g) = C \int_{G(\mathbb{Q})\backslash G(\mathbf{A})/KK_\infty} [f_\alpha \wedge g] \quad \forall g \in {}_K Z^q_{(2),\sigma}. \tag{6.2.1} $$

Since f_α is rapidly decreasing, hence $L_2 \pmod{Z_G(\mathbf{A})}$, the lemma follows from (6.2.1).

In order to prove Theorem (5.3), it thus remains to show that every element of the image of $\bar{\lambda}$ is represented by an element of ${}_K L^{\cdot}_{(2),\sigma,\mathrm{disc}}$. We write ${}_K \bar{Z}^q_{(2),\sigma} = \mathrm{Ker}(\bar{\lambda}) \oplus W$, where W is the orthogonal complement of $\mathrm{Ker}(\bar{\lambda})$. We will be done if we can show that $W \subset {}_K L^{\cdot}_{(2),\sigma,\mathrm{disc}}$. Now the Hecke algebra $\mathscr{H}(G(\mathbf{A}^f), K)$ acts on the complexes ${}_K C^{\cdot}_{(2),\sigma}$ and ${}_K C^{\cdot}_{\mathrm{si},\sigma}$; the action is compatible with the inclusion ${}_K C^{\cdot}_{(2),\sigma} \subset {}_K C^{\cdot}_{\mathrm{si},\sigma}$, and the action on ${}_K C^{\cdot}_{(2),\sigma}$ is continuous in the L_2 topology. Thus $\mathrm{Ker}(\bar{\lambda})$ and W are $\mathscr{H}(G(\mathbf{A}^f), K)$-modules. Since W is finite dimensional, it consists of $\mathscr{H}(G(\mathbf{A}^f), K)$-finite vectors. Theorem (5.3) now follows from Lemma (6.1).

(**6.3**) The proof of Lemma (6.1) is based on the structure of the continuous spectrum, given by Langlands' theory of Eisenstein series, and is motivated by the proof by Borel and Garland [10] of the analogous theorem for archimedean primes. Before arriving at Eisenstein series, we have to discuss parabolic subgroups, Hecke algebras, and direct integrals.

Let B be a minimal rational parabolic subgroup of G, and let $A \subset B$ be a split component, $M \subset B$ the centralizer in G of A, and W the Weyl group of G relative to A. Let p be a rational prime such that $K_p =_{\text{def}} K \cap G(\mathbb{Q}_p)$ is a special maximal compact subgroup in the sense of [51]. Thus $\mathscr{H} = \mathscr{H}(G_p, K_p)$ is commutative. Let $B' \subset B$ be a minimal $\mathbb{Q}_p$-rational parabolic subgroup of $G(\mathbb{Q}_p)$, let $A' \supset A$ be a split component of B', $M' = Z_{G(\mathbb{Q}_p)}(A')$, and $W = W(G(\mathbb{Q}_p), A')$.

Let P be a standard $\mathbb{Q}_p$-rational parabolic subgroup of $G(\mathbb{Q}_p)$ and let $M \supset M'$ be the standard Levi component. Define $M^0 = \{m \in M \mid |\chi(m)|_p = 1$ for all algebraic characters $\chi\colon M \to \mathbb{G}_m\}$, $\Lambda(M) = M/M^0$. The *Satake isomorphism* [41] is a map $S\colon \mathscr{H} \cong \mathbb{C}[\Lambda(M')]^{W'}$.

Let P, M be as above, and let (σ, V_σ) be an irreducible admissible representation of M. Let $P_1 \supset P$ be another standard parabolic, M_1 its standard Levi component. We view σ as a representation of P in which $R_u(P)$ acts trivially. Define $I(M, M_1, \sigma)$ to be the space of locally constant V_σ-valued functions f on M_1 satisfying

$$f(pg) = \delta_P^{1/2}(p)\sigma(p)f(g) \quad \text{for } p \in P,\ g \in M_1,$$

where δ_P is the modulus character of P. The group M_1 acts on $I(M, M_1, \sigma)$ by right translation; the resulting representation $\pi(\sigma)$ is unitary if σ is.

A character of M is said to be *unramified* if it is trivial on M^0. Let χ be an unramified character of M'. Let $I(\chi) = I(M', G(\mathbb{Q}_p), \chi)$. The algebra $\mathscr{H}(G(\mathbb{Q}_p))$ of locally constant compactly supported functions on $G(\mathbb{Q}_p)$ acts naturally on $I(\chi)$; the subalgebra $\mathscr{H}$ of $\mathscr{H}(G(\mathbb{Q}_p))$ fixes the one-dimensional subspace $W(\chi)$ of K_p-fixed vectors. The action of $\mathscr{H}$ on $W(\chi)$ is given by

$$\text{(6.3.1)} \qquad \pi(\chi)(\varphi)(v) = S(\varphi)(\chi)\cdot v \quad \text{for } \varphi \in \mathscr{H},\ v \in W(\chi).$$

Now let P, M be as above, and let σ be an irreducible unitary representation of M which arises as a subquotient of $I(M', M, \chi)$ for some unramified character χ of M'. We assume σ contains a vector fixed under $K_p \cap M$. The group $\Lambda(M)$ is free abelian; thus the space $X(M)$ of unramified unitary characters of M is naturally a compact torus. For any

$\nu \in X(M)$, let ν' denote its restriction to M'. By transitivity of induction, the representation $I(M, G(\mathbb{Q}_p), \sigma \otimes \nu)$ is a unitary subquotient of $I(\chi \otimes \nu')$, and contains a nontrivial K_p-fixed subspace $W(\sigma, \nu)$. Since K_p acts semisimply on $I(\chi \otimes \nu')$, $W(\sigma, \nu)$ and $W(\chi \otimes \nu')$ are naturally isomorphic as modules over $\mathscr{H}$.

Let $\tilde{X}(M)$ be the universal covering space of $X(M)$ and $r\colon \tilde{X}(M) \to X(M)$ the natural map. For σ as above and $\tilde{\nu} \in \tilde{X}(M)$, let

$$I(M, G(\mathbb{Q}_p), \sigma \otimes \tilde{\nu}) = I(M, G(\mathbb{Q}_p), \sigma \otimes r(\tilde{\nu})).$$

Let D be a connected open subset of $\tilde{X}(M)$, and let $I_D(M, G(\mathbb{Q}_p), \sigma)$ be the direct integral

$$I_D(M, G(\mathbb{Q}_p), \sigma) = \int_D^{\oplus} I(M, G(\mathbb{Q}_p), \sigma \otimes \nu)\, dx,$$

where dx is a Haar measure on $\tilde{X}(M)$. Let $\pi_D(\sigma)$ denote the representation of $G(\mathbb{Q}_p)$ on $I_D(M, G(\mathbb{Q}_p), \sigma)$. We have an isometric isomorphism of $\mathscr{H}$-modules

$$I_D(M, G(\mathbb{Q}_p), \sigma)^{K_p} \cong \int_D^{\oplus} W(\chi \otimes \nu')\, dx. \tag{6.3.2}$$

It follows from (6.3.1) and (6.3.2) that there exists $\varphi \in \mathscr{H}$ such that $\pi_D(\sigma)(\varphi)$ has continuous spectrum on $I_D(M, G(\mathbb{Q}_p), \sigma)^{K_p}$. In particular, $I_D(M, G(\mathbb{Q}_p), \sigma)^{K_p}$ contains no $\mathscr{H}$-finite vectors.

Now Langlands' theory of Eisenstein series [33, Appendix 2]; see also [10, p. 325] for a clear statement in the context of arithmetic subgroups of $G(\mathbb{R})$ implies that the space ${}_K L^{\bullet}_{(2), \sigma, \mathrm{cont}}$ is isomorphic as an $\mathscr{H}$-module to a countable Hilbert direct sum of spaces of the form $I_D(M, G(\mathbb{Q}_p), \sigma)^{K_p}$. Here (M, σ) runs through a subset of the set of pairs

{standard Levi components M of (the p-adic points of)
$\mathbb{Q}$-rational standard parabolic subgroups,
irreducible unitary representations σ of M}

modulo a certain equivalence relation, and D is a specific open subset of $\tilde{X}(M)$. It follows from the preceding argument that none of the spaces $I_D(M, G(\mathbb{Q}_p), \sigma)^{K_p}$ contains $\mathscr{H}$-finite vectors. This completes the proof of the lemma, and hence of Theorem (5.3).

7. Rationality criteria for cusp forms

One of the main consequences of our work up to this point has been the definition of rational structures over number fields on spaces of cusp forms of $\bar{\partial}$-cohomology type by identifying these forms with certain kinds of cohomology classes. In this section and the next, we propose a criterion which, in certain cases, permits us to determine which cusp forms actually define cohomology classes defined over $\bar{\mathbb{Q}}$. The criterion generalizes Shimura's method for determining rationality of holomorphic modular forms by studying their values at CM points [49].

(7.1) We let (G, X) have its usual meaning. Let $(G^{\#}, X^{\#}) \subset (G, X)$ be another basic pair. We assume our point $h \in X$ is actually in $X^{\#}$, and let $K_{\infty}^{\#} \subset K_{\infty}$ be its stabilizer in $G^{\#}(\mathbb{R})$; define $\mathfrak{P}_h^{\#}$, $\mathfrak{p}^{\#,-}$ in the obvious way. Let (π, V) (resp. $(\pi^{\#}, V^{\#})$) be an irreducible representation of $G(\mathbb{R})^0$ (resp. $G^{\#}(\mathbb{R})^0$), unitary on G_0 (resp. $G_0^{\#}$). Thus V and $V^{\#}$ are taken to be Hilbert spaces, with inner products $(\cdot, \cdot)_V$ and $(\cdot, \cdot)_{V^{\#}}$, respectively. Assume $\pi^{\#}$ is isomorphic to a closed direct factor of $\pi|_{G^{\#}(\mathbb{R})^0}$, and identify $V^{\#}$ with a closed subspace of V by a homomorphism $U: V^{\#} \to V$. Let $V(\pi^{\#}) \subset V$ denote the $\pi^{\#}$-isotypic subspace, and let $p: V \to V(\pi^{\#})$ be the orthogonal projection, p_U the orthogonal projection on $U(V^{\#})$.

As usual, we let V_0 denote the space of K_{∞}-finite vectors of V, and define $V_0^{\#}$ likewise. Let $T: V_0 \to \mathscr{A}_0(G)$ be a unitary homomorphism of $(\mathfrak{g}, K_{\infty})$-modules. Let $L_{2,0}(G)$ (resp. $L_{2,0}(G^{\#})$) be the completion of $\mathscr{A}_0(G)$ (resp. $\mathscr{A}_0(G^{\#})$) with respect to the L_2 metric (5.1.1).

Let V_T be the closure of $\mathrm{Im}(T)$ in $L_{2,0}(G)$. Then $V_T \cong V$. We assume $Z_G(\mathbb{R})$ acts on V by the character χ; then for any $f \in V_T$ the function $f \otimes \xi_{\chi}^{-1}$, defined as in (5.1), is square integrable on $G(\mathbb{Q})\backslash G(\mathbf{A})$. Let $V_T^{\#} \subset V_T$ be the closed subspace corresponding to $U(V^{\#}) \subset V$; let $\mathrm{pr}: V_T \to V_T^{\#}$ be the orthogonal projection.

We assume that the restriction of $\pi^{\#}$ to $G_0^{\#}$ is a representation of the discrete series. Thus, if $v, w \in V_0^{\#}$, then the matrix coefficient

$$c_{v,w}(h) \underset{\mathrm{def}}{=} (\pi^{\#}(h)v, w)_{V^{\#}}, \qquad h \in G_0^{\#},$$

is square integrable as a function on $G_0^{\#}$.

In general, the elements of $V_T^{\#} \subset V_T$ are not automorphic forms on $G(\mathbb{Q})\backslash G(\mathbf{A})$, nor are they even K_{∞}-finite. Under certain hypotheses, however, we can say something nontrivial about the elements of $\mathrm{pr} \circ T(V_0)$:

(7.2) Lemma. *Suppose $\pi^{\#}$ is an integrable discrete series representation of $G_0^{\#}$; i.e., $c_{v,w} \in L_1(G_0^{\#})$ for all $v, w \in V_0^{\#}$. Then, for any $f \in V_0$ and* *any* $T \in \operatorname{Hom}_{(\mathfrak{g}, K_\infty)}(V_0, \mathscr{A}_0(G))$, *the function* $\operatorname{pr} \circ T(f)$ *is a* C^∞*-function on* $G(\mathbb{Q})\backslash G(\mathbf{A})$ *which is bounded modulo* $Z_G(\mathbf{A})$; *i.e.,* $\operatorname{pr} \circ T(f) \otimes \xi_\chi^{-1}$ *is a bounded function.*

Remark. We thank Roger Howe for suggesting that such a statement should be true.

Proof. The argument is routine. Since f is K_∞-finite and $\operatorname{pr} \circ T$ commutes with the action of $G^\#(\mathbb{R})^0$, $\operatorname{pr} \circ T(f)$ is clearly a $K_\infty^\#$-finite vector. Let $v = \operatorname{pr} \circ T(f) \in (V_T^\#)_0$, which we identify with $V_0^\#$, and let c be the matrix coefficient $c_{v,v}$ of $\pi^\#$, which we extend to $G^\#(\mathbb{R})$ by defining it to be zero off $G^\#(\mathbb{R})^0$. By twisting with ξ_χ^{-1}, we may assume that the central character χ of π is unitary; then the matrix coefficients of $\pi^\#$ belong to $L^1(G^\#(\mathbb{R})) \cap L^2(G^\#(\mathbb{R}))$. Choose a Haar measure dh on $G^\#(\mathbb{R})$, and let δ be the formal degree of $\pi^\#$, defined in terms of dh [31, p. 284]. Then the operator $C = \delta\overline{\pi^\#(c)}$, defined as usual by

$$(7.2.1) \qquad Cw = \delta \int_{Z_{G^\#}(\mathbb{R})\backslash G^\#(\mathbb{R})} \overline{c(h)}\pi^\#(h) w \, dh,$$

is well defined and coincides with orthogonal projection on the subspace $\mathbb{C}v \subset V_0^\#$ (this is essentially the definition of δ).

Write $G' = Z_{G^\#}(\mathbb{R})\backslash G^\#(\mathbb{R})$. Define $C\colon V_T \to V_T$ by formula (7.2.1):

$$(7.2.2) \qquad (CF)(g) = \delta \int_{G'} \overline{c(h)} F(gh)\, dh, \qquad F \in V_T, \ g \in G(\mathbf{A}).$$

Note that $\overline{c(h)} = c(h^{-1})$. We claim that C is a bounded operator on V_T. In fact, let $g \mapsto R_g$ be the right regular representation of $G(\mathbb{R})$ on $L_{2,0}(G)$, and let $(\cdot, \cdot)_{(2)}$ be the L_2 norm on $L_{2,0}(G)$. If $F \in V_T$, then

$$\begin{aligned}
&(CF, CF)_{(2)} \\
&= \delta^2 \int_{Z_G(\mathbf{A})G(\mathbb{Q})\backslash G(\mathbf{A})} \left(\int_{G'} \int_{G'} \overline{c(h_1)} F(gh_1)\, dh_1 \overline{\overline{c(h_2)} F(gh_2)\, dh_2} \right) dg \\
&= \delta^2 \int_{G'} \int_{G'} \overline{c(h_1)} c(h_2) (R_{h_1} F, R_{h_2} F)_{(2)}\, dh_1\, dh_2 \\
&\hspace{20em} \text{(by Fubini's Theorem)} \\
&\leq \delta^2 \|R_{h_1} F\|_{(2)} \|R_{h_2} F\|_{(2)} \int_{G'} \int_{G'} \overline{c(h_1)} c(h_2)\, dh_1\, dh_2 \\
&\hspace{20em} \text{(by Schwarz' inequality)} \\
&\leq \delta^2 M^2 (F, F)_{(2)},
\end{aligned}$$

where M is the L^1 norm of c.

Moreover, C is a selfadjoint operator on V_T. In fact, if $F, F' \in V_T$, then

$$\begin{aligned}\delta(CF, F')_{(2)} &= \int_{Z_G(\mathbf{A})G(\mathbb{Q})\backslash G(\mathbf{A})} \int_{G'} \overline{c(h)} F(gh) \overline{F'(g)}\, dh\, dg \\ &= \int_{Z_G(\mathbf{A})G(\mathbb{Q})\backslash G(\mathbf{A})} \int_{G'} \overline{c(h)} F(g) \overline{F'(gh^{-1})}\, dh\, dg \\ &= \int_{Z_G(\mathbf{A})G(\mathbb{Q})\backslash Q(\mathbf{A})} \int_{G'} F(g) \overline{c(h^{-1})}\, \overline{F'(gh)}\, dh\, dg = \delta(F, CF')_{(2)}.\end{aligned}$$

On the other hand, $C(V_T^{\#}) \subset V_T^{\#}$, and $C|_{V_T^{\#}}$ is the orthogonal projection on $\mathbb{C}v$. Then we easily see that $C \circ T(f) = \mathrm{pr} \circ T(f)$. It now follows immediately from (7.2.2) that, for any $f \in V_0$, $\mathrm{pr} \circ T(f)$ is a C^∞ function and is bounded above by $\delta \cdot M \cdot N$, where N is any upper bound for $|T(f)(g)|$ on $G(\mathbb{Q})\backslash G(\mathbf{A})$.

For any $\gamma \in G(\mathbf{A}^f)$, let $T_\gamma(f) \in \mathscr{A}_0(G)$ be the function $g \mapsto T(f)(g\gamma)$; let $R_\gamma(f) = \mathrm{pr} \circ T_\gamma(f) \in C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))$.

(7.3) Corollary. *Under the hypotheses of Lemma* (7.2), *the restriction* F_γ *of* $R_\gamma(f)$ *to* $G^{\#}(\mathbb{Q})\backslash G^{\#}(\mathbf{A})$ *is a cusp form for any* $\gamma \in G(\mathbf{A}^f)$.

Proof. By twisting with ξ_χ^{-1}, we may assume that the central character χ of π is unitary. It then follows from Lemma (7.2) that F_γ is bounded, hence square integrable, on $G^{\#}(\mathbb{Q})\backslash G^{\#}(\mathbf{A})$. On the other hand, $R_\gamma(f)$ is contained in the $(\mathfrak{g}^{\#}, K_\infty^{\#})$-submodule $W_\gamma = T_\gamma(V_0^{\#}) \subset C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))$. If we denote by Res the restriction map

$$C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A})) \to C^\infty(G^{\#}(\mathbb{Q})\backslash G^{\#}(\mathbf{A})),$$

we thus have $F_\gamma = \mathrm{Res}\, R_\gamma(f) \subset \mathrm{Res}(W_\gamma)$; the latter space is either trivial or isomorphic to $V_0^{\#}$ as a $(\mathfrak{g}^{\#}, K_\infty^{\#})$-module. It follows that F_γ is $Z(\mathfrak{g}_{\mathbb{C}}^{\#})$-finite and $K_\infty^{\#}$-finite, and thus is an automorphic form. But $V_0^{\#}$ is a discrete series representation, hence a tempered $(\mathfrak{g}^{\#}, K_\infty^{\#})$-module. The corollary now follows from Wallach's Theorem (5.3.1).

(7.3.1) For any $\gamma \in G(\mathbf{A}^f)$, Corollary (7.3) thus defines a homomorphism, depending on our intertwining operator $U: V^{\#} \to V$:

$$\beta_\gamma = \beta_{\gamma, U}\colon \mathrm{Hom}_{(\mathfrak{g}, K_\infty)}(V_0, \mathscr{A}_0(G)) \to \mathrm{Hom}_{(\mathfrak{g}^{\#}, K_\infty^{\#})}(V_0^{\#}, \mathscr{A}_0(G^{\#})),$$
$$T \mapsto (w \mapsto \mathrm{Res}\, R_\gamma(f)), \qquad w \in V_0^{\#},$$

where f is any element of V_0 such that $p_U(f) = w$. The existence of such an f follows from the fact that V_0 is dense in V. We write

$$\mathscr{A}_0(G, \pi) = \mathrm{Hom}_{(\mathfrak{g}, K_\infty)}(V_0, \mathscr{A}_0(G)),$$
$$\mathscr{A}_0(G^\#, \pi^\#) = \mathrm{Hom}_{(\mathfrak{g}^\#, K_\infty^\#)}(V_0^\#, \mathscr{A}_0(G^\#)).$$

(7.4) Theorem. *We retain the notation and hypotheses of Lemma* (7.2). *Assume T is nontrivial. Let $f \in V_0$ be a vector such that $0 \neq p(v) \in V_0^\#$, and let $F = T(f) \in \mathscr{A}_0(G)$. Let χ be the central character of π, and define ξ_χ as in* (5.1). *Write $\Gamma = Z_{G^\#}(\mathbf{A})G^\#(\mathbb{Q})\backslash G^\#(\mathbf{A})$. Then there exist $\gamma \in G(\mathbf{A}^f)$, a homomorphism $T' \colon V_0^\# \to \mathscr{A}_0(G^\#)$ of $(\mathfrak{g}^\#, K_\infty^\#)$-modules, and a cusp form $F' \in T'(V_0^\#)$ such that*

(i) $\beta_\gamma(T) \neq 0$, *and*

(ii) $\int_\Gamma F(h\gamma)\overline{F'(h)}\xi_\chi(h)^{-2}\,dh = \int_\Gamma \beta_\gamma(T)(p_U(f))\overline{F'(h)}\xi_\chi(h)^{-2}\,dh \neq 0.$

(7.4.1) Remark. We emphasize that at no point in the proofs of this or the preceding results do we use the complex structures of X and $X^\#$. In particular, (7.2)–(7.4) are valid whenever $G^\#$ and $K_\infty^\#$ are of equal rank.

Proof. We lose no generality by assuming $\xi_\chi \equiv 1$. Our hypotheses imply that the function $\mathrm{pr}(F) \in C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))$ is not identically zero. Thus there exists $\gamma \in G(\mathbf{A}^f)$ such that $F_\gamma(h)$, defined as in Corollary (7.3), is not identically zero as a function on $G^\#(\mathbf{A})$. In fact, $G(\mathbb{Q})$ is dense in $G(\mathbb{R})$ [17], hence the image of $G^\#(\mathbf{A}) \cdot G(\mathbf{A}^f)$ is dense in $G(\mathbb{Q})\backslash G(\mathbf{A})$.

We let $T' = \beta_\gamma(T)$, in the notation of (7.3.1). Then F_γ is a nonzero element of $\mathrm{Im}(T')$. We take $F' = F_\gamma = \mathrm{Res}(R_\gamma)$, as in the proof of (7.3). It remains to be shown that

$$\int_\Gamma F(h\gamma)\overline{R_\gamma(h)}\,dh = \int_\Gamma R_\gamma(h)\overline{R_\gamma(h)}\,dh. \tag{7.4.2}$$

Recall that $R_\gamma = \mathrm{pr} \circ T_\gamma(f) = C(T_\gamma(f))$, where C is given by the formula (7.2.2). By replacing T by T_γ, we may assume $\gamma = 1$; we write $R = R_\gamma$. Recall also that $C(R) = R$. Now

(7.4.3)

$$\begin{aligned}\int_\Gamma F(h)\overline{R(h)}\,dh &= \int_\Gamma F(h)\overline{CR(h)}\,dh = \delta\int_\Gamma F(h)\int_{B'} c(h_1)\overline{R(hh_1)}\,dh_1\,dh\\ &= \delta\int_\Gamma\int_{G'} F(h)c(h^{-1}h_1)\overline{R(h_1)}\,dh_1\,dh\\ &= \delta\int_\Gamma\int_{Z_{G^\#}(\mathbf{A})\backslash G^\#(\mathbf{A})} F(h)c_{\mathbf{A}}(h^{-1}h_1)\overline{R(h_1)}\,dh_1\,dh.\end{aligned}$$

Here $c_{\mathbf{A}} \in L_1(G^{\#}(\mathbf{A}))$ is defined as follows: Choose a level subgroup $K^{\#} \subset G^{\#}(\mathbf{A}^f)$ such that F is right invariant under $K^{\#}$, and define $c_{\mathbf{A}}$ on $g_\infty \cdot g^f \in G^{\#}(\mathbb{R}) \cdot G^{\#}(\mathbf{A}^f)$ by the formula

$$c_{\mathbf{A}}(g_\infty \cdot g^f) = \begin{cases} (\operatorname{vol} K^{\#})^{-1} c(g_\infty) & \text{if } g^f \in K^{\#}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $P_c(h, h_1) = \sum_{\alpha \in G(\mathbb{Q})} c_{\mathbf{A}}(h^{-1}\alpha h_1)$. The series P_c converges absolutely, since $c_{\mathbf{A}} \in L_1(G^{\#}(\mathbf{A}))$. Continuing as above, we find that the left-hand side of (7.4.3) is equal to

$$(7.4.4)\quad \begin{aligned} &\delta \int_\Gamma \int_\Gamma F(h) P_c(h, h_1) \overline{R(h_1)}\, dh_1\, dh \\ &\qquad = \delta \int_\Gamma \left(\int_\Gamma F(h) P_c(h, h_1)\, dh \right) \overline{R(h_1)}\, dh_1 \end{aligned}$$

(by Fubini's theorem).

The inner integral in (7.4.4) is equal to

$$\begin{aligned} \int_\Gamma F(h) P_c(h, h_1)\, dh &= \int_{Z_{G^{\#}}(\mathbf{A}) \backslash G^{\#}(\mathbf{A})} F(h) c_{\mathbf{A}}(h^{-1} h_1)\, dh \\ &= \int_{Z_{G^{\#}}(\mathbf{A}) \backslash G^{\#}(\mathbf{A})} \overline{c_{\mathbf{A}}(h_1^{-1} h)} F(h)\, dh \\ &= \int_{G'} \overline{c(h_1^{-1} h)} F(h)\, dh \\ &= \delta^{-1} (CF)(h_1). \end{aligned}$$

Combining this with (7.4.3), we obtain (7.4.2) and hence the theorem.

(7.5) We now apply the above results to the cohomology groups we have been studying. Choose irreducible unitary representations σ and $\sigma^{\#}$ of K_∞ and $K_\infty^{\#}$, respectively. We make the following hypotheses:

(7.5.1) $\sigma^{\#}$ is a direct factor of $\sigma|_{K_\infty^{\#}}$; let $p_{\sigma, \sigma^{\#}}: V_\sigma \to V_{\sigma^{\#}}$ denote the orthogonal projection.

(7.5.2) $\dim H^q(\mathfrak{P}_h, K_\infty, V_0 \otimes V_\sigma) = \dim H^q(\mathfrak{P}_h^{\#}, K_\infty^{\#}, V_0^{\#} \otimes V_{\sigma^{\#}}) = 1$.

(7.5.3) The orthogonal projection $p \otimes p_{\sigma, \sigma^{\#}}: V \otimes V_\sigma \to V(\pi^{\#}) \otimes V_{\sigma^{\#}}$ induces a nontrivial homomorphism on $\bar{\partial}$-cohomology:

$$H^q(\mathfrak{P}_h, K_\infty, V_0 \otimes V_\sigma) \to H^q(\mathfrak{P}_h^{\#}, K_\infty^{\#}, V(\pi^{\#})_0 \otimes V_{\sigma^{\#}}).$$

We can actually strengthen condition (7.5.3). By Proposition (4.5), we have $H^q(\mathfrak{P}_h, K_\infty, V_0 \otimes V_\infty) = \operatorname{Hom}_{K_\infty}(\Lambda^q(\mathfrak{p}^-) \otimes V_\sigma^*, V_0)$. By (7.5.2) there is thus, up to scalars, a unique nontrivial $h \in \mathrm{Hom}_{K_\infty}(\Lambda^q(\mathfrak{p}^-) \otimes V_\sigma^*, V_0)$. Let $W = \mathrm{Im}(h)$, and let $W(\pi^\#)$ be the closure in V of the $(\mathfrak{g}^\#, K_\infty^\#)$-submodule of V_0 generated by W. Then

(7.5.3)′ The orthogonal projection $p \otimes p_{\sigma,\sigma^\#}: V \otimes V_\sigma \to V(\pi^\#) \otimes V_{\sigma^\#}$ induces a nontrivial homomorphism on $\bar{\partial}$-cohomology:

$$H^q(\mathfrak{P}_h, K_\infty, V_0 \otimes V_\sigma) \to H^q(\mathfrak{P}_h^\#, K_\infty^\#, W(\pi^\#)_0 \otimes V_{\sigma^\#}).$$

Let

$$H^q(\pi \otimes \sigma) = H^q(\mathfrak{P}_h, K_\infty, V_0 \otimes V_\sigma), \; H^q(\pi^\# \otimes \sigma^\#) = H^q(\mathfrak{P}_h^\#, K_\infty^\#, V^\# \otimes V_{\sigma^\#}).$$

Note that

$$\begin{aligned} H^q(\mathfrak{P}_h^\#, K_\infty^\#, W(\pi^\#)_0 \otimes V_{\sigma^\#}) &= H^q(\pi^\# \otimes \sigma^\#) \otimes \mathrm{Hom}_{(\mathfrak{g}^\#, K_\infty^\#)}(V_0^\#, W(\pi^\#)_0) \\ &\cong \mathrm{Hom}_{(\mathfrak{g}^\#, K_\infty^\#)}(V_0^\#, W(\pi^\#)_0), \end{aligned}$$

by (7.5.2). Since $W(\pi^\#)_0$ is a unitary $\mathfrak{g}^\#$-module generated by the finite-dimensional $K_\infty^\#$-module W, it follows easily from Frobenius reciprocity that $\mathrm{Hom}_{(\mathfrak{g}^\#, K_\infty^\#)}(V_0^\#, W(\pi^\#)_0)$ is finite dimensional. Up to multiplication by a scalar of absolute value 1, there is thus a unique unitary imbedding $U_0: V^\# \to V$ of Hilbert spaces with $G^\#$-action such that the orthogonal projection $p_{U_0} \otimes p_{\sigma,\sigma^\#}: V \otimes V_\sigma \to V_0^\# \otimes V_\sigma$ induces a nontrivial homomorphism

$$H(\sigma, \sigma^\#) = H(p_{U_0} \otimes p_{\sigma,\sigma^\#}): H^q(\pi \otimes \sigma) \xrightarrow{\sim} H^q(\pi^\# \otimes \sigma^\#).$$

We fix U_0 in what follows.

Recall (Proposition (4.5)) that $H^q(\pi \otimes \sigma) \cong \mathrm{Hom}_{K_\infty}(\Lambda^q(\mathfrak{p}^-), V_0 \otimes V_\sigma)$ for unitary V_0; the analogous formula holds for $H^q(\pi^\# \otimes \sigma^\#)$. The hypothesis (7.5.3) is the assertion that the composite map

$$\begin{aligned} \mathrm{Hom}_{K_\infty}(\Lambda^q(\mathfrak{p}^-), V_0 \otimes V_\sigma) &\to \mathrm{Hom}_{K_\infty^\#}(\Lambda^q(\mathfrak{p}^{\#,-}) V_0 \otimes V_\sigma) \\ &\to \mathrm{Hom}_{K_\infty^\#}(\Lambda^q(\mathfrak{p}^{\#,-}), V_0^\# \otimes V_{\sigma^\#}) \end{aligned}$$

is nontrivial, where the first arrow is induced by the inclusion $\mathfrak{p}^{\#,-} \subset \mathfrak{p}^-$, and the second arrow by $p_{U_0} \otimes p_{\sigma,\sigma^\#}$.

For any $\gamma \in G(\mathbf{A}^f)$ we let $\alpha_\gamma = \alpha_\gamma(\sigma, \sigma^\#)$ be the homomorphism

$$H(\sigma, \sigma^\#) \otimes \beta_{\gamma, U_0}: H^q(\pi \otimes \sigma) \otimes \mathscr{A}_0(G, \pi) \to H^q(\pi^\# \otimes \sigma^\#) \otimes \mathscr{A}_0(G^\#, \pi^\#).$$

By Lemma (5.2.3), we may regard α_γ as a homomorphism

$$\alpha_\gamma : \mathscr{H}^q_{\mathrm{cusp},\sigma} \to \mathscr{H}^q_{\mathrm{cusp},\sigma^\#}.$$

Here α_γ factors through the direct factor $H^q(\pi\otimes\sigma)\otimes\mathscr{A}_0(G,\pi)$ of $\mathscr{H}^q_{\mathrm{cusp},\sigma}$, and $\mathrm{Im}(\alpha_\gamma) \subset H^q(\pi^\# \otimes \sigma^\#) \otimes \mathscr{A}_0(G^\#, \pi^\#)$.

Now let $\mathscr{V} = \check{E}_\sigma$ and $\mathscr{V}^\# = \check{E}_{\sigma^\#}$, in the notation of (2.1). Then the pullback of the automorphic vector bundle $[\mathscr{V}]$ to $M(G^\#, X^\#)$ contains $[\mathscr{V}^\#]$ as a direct factor. Proposition (2.6) thus provides us with a homomorphism

$$\psi: \bar{H}^q([\mathscr{V}]) = \bar{H}^q(M(G,X),[\mathscr{V}]) \to \bar{H}^q([\mathscr{V}^\#]) = \bar{H}^q(M(G^\#,X^\#),[\mathscr{V}^\#]).$$

For $\gamma \in G(\mathbf{A}^f)$, we let $\psi_\gamma : \bar{H}^q([\mathscr{V}]) \to \bar{H}^q([\mathscr{V}^\#])$ denote $\psi \circ t^*_\gamma$, where $\gamma \mapsto t^*_\gamma$ is the action of $G(\mathbf{A}^f)$ given by Proposition (2.6). Consider the diagram:

$$\text{(7.5.4)} \qquad \begin{array}{ccc} \mathscr{H}^q_{\mathrm{cusp},\sigma} & \xrightarrow{\mathrm{cl}} & \bar{H}^q([\mathscr{V}]) \\ {\scriptstyle\alpha_\gamma}\downarrow & & \downarrow{\scriptstyle\psi_\gamma} \\ \mathscr{H}^q_{\mathrm{cusp},\sigma^\#} & \xrightarrow{\mathrm{cl}} & \bar{H}^q([\mathscr{V}^\#]) \end{array}$$

Here the homomorphism cl is defined as in (3.5).

(7.5.5) Lemma. *Suppose the representation* $\sigma^\#$ *and the integer* q *satisfy condition* (a) *of Corollary* (5.3.2). *Then diagram* (7.5.4) *is commutative.*

Proof. Since the upper horizontal arrow is a homomorphism of $G(\mathbf{A}^f)$-modules, we may assume $\gamma = 1$; we write $\alpha = \alpha_1$ and $\psi = \psi_1$. Now the commutative diagram in Lemma (4.2.2) is clearly functorial with respect to the inclusion $(G^\#, X^\#) \subset (G, X)$. It follows that the diagram

(7.5.5.1)

$$\begin{array}{ccc} C^\cdot(\mathfrak{P}_h, K_\infty, \mathscr{A}_0(G)\otimes V_\sigma) & \longrightarrow & C^\cdot(\mathfrak{P}_h, K_\infty, C^\infty(G(\mathbb{Q})\backslash G(\mathbf{A}))\otimes V_\sigma) \\ \downarrow{\scriptstyle R} & & \downarrow{\scriptstyle R'} \\ C^\cdot_{\mathrm{rd},\sigma^\#} & \longrightarrow & C^\cdot(\mathfrak{P}^\#_h, K^\#_\infty, C^\infty(G^\#(\mathbb{Q})\backslash G^\#(\mathbf{A}))\otimes V_\sigma) \end{array}$$

is commutative, where R and R' are given by restriction of functions.

Now under condition (a) of Corollary (5.3.2), the lower horizontal arrow in (7.5.4) is an isomorphism (condition (b) is automatic); our second hypothesis implies that

$$\mathscr{H}^q_{\mathrm{cusp},\sigma^\#} = H^q(\pi^\# \otimes \sigma^\#)\otimes\mathscr{A}_0(G^\#, \pi^\#) \cong \mathscr{A}_0(G^\#, \pi^=\#).$$

Thus, for every $\varphi \in H^q(\mathfrak{P}_h, K_\infty, \mathscr{A}_0(G)\otimes V_\sigma)$, $R(\varphi)$ has a unique $\Box_{q,\sigma^\#}$-harmonic representative $\varphi^\#$ in $\mathrm{cl}(\mathscr{A}_0(G^\#, \pi^\#))$. It remains to show that if $\varphi \in H^q(\pi\otimes\sigma)\otimes\mathscr{A}_0(G, \pi)$, then $\varphi^\# = \alpha(\varphi)$.

We may choose φ to be an element of $\mathrm{Hom}_{K_\infty}(\Lambda^q(\mathfrak{p}^-), T(V_0)\otimes V_\sigma)$ for some $T \in \mathscr{A}_0(G, \pi)$. Restriction of φ to $\Lambda^q(\mathfrak{p}^{\#,-})$, followed by restriction of functions from $G(\mathbf{A})$ to $G^\#(\mathbf{A})$, defines an element of $\mathrm{Hom}_{K^\#_\infty}(\Lambda^q(\mathfrak{p}^{\#,-}), \mathrm{Res}(T(V_0))\otimes V_{\sigma^\#})$, where Res is restriction of functions; by (7.5.5.1) this element is just $R(\varphi)$. Now Proposition (4.4.3) implies that $\varphi^\#$ is obtained from $R(\varphi)$ by taking the orthogonal projection of $\mathrm{Res}(T(V_0))$ on the space of eigenfunctions for $C_\mathfrak{g}$ with eigenvalue $c_{\Lambda^\#}$, where $\Lambda^\#$ is the highest weight of $\sigma^\#$. By hypothesis (a) of Corollary (5.3.2), this defines the same element of $\mathrm{Hom}_{K^\#_\infty}(\Lambda^q(\mathfrak{p}^{\#,-}), \mathscr{A}_0(G^\#)\otimes V_{\sigma^\#})$ as is obtained by projecting $\mathrm{Res}(T(V_0))$ on the $\pi^\#$-isotypic component of $\mathscr{A}_0(G^\#)$. In view of the remarks following (7.5.3), this completes the proof.

Now suppose the homogeneous vector bundles $\mathscr{V}$ and $\mathscr{V}^\#$ and the homomorphism $\mathscr{V}|_{\check{M}(G^\#, X^\#)} \to \mathscr{V}^\#$ are all defined over the extension $k^\#$ of $E(G^\#, X^\#)$. Then for all $\gamma \in G(\mathbf{A}^f)$, the homomorphism $\psi_\gamma : \bar{H}^q([\mathscr{V}]) \to \bar{H}^q([\mathscr{V}^\#])$ is also rational over $k^\#$. It follows that, for any extension L of $k^\#$, if $\varphi \in \mathscr{H}^q_{\mathrm{cusp},\sigma}$ is such that $\mathrm{cl}(\varphi)$ is an L-rational class in $\bar{H}^q([\mathscr{V}])$, then $\psi_\gamma(\mathrm{cl}(\varphi)) = \mathrm{cl}\circ\alpha_\gamma(\varphi)$ is an L-rational class in $\bar{H}^q([\mathscr{V}^\#])$. The following theorem is a partial converse to this statement.

(7.6) Theorem. *Let (π, V) (resp. $(\pi^\#, V^\#)$) be an irreducible unitary representation of $G(\mathbb{R})^0$ (resp. $G^\#(\mathbb{R})^0$) such that $\pi^\#$ is a closed direct factor of $\pi|_{G^\#(\mathbb{R})^0}$. Let (σ, V_σ) (resp. $(\sigma^\#, V^\#_\sigma)$) be an irreducible unitary representation of K_∞ (resp. $K^\#_\infty$) such that $\sigma^\#$ is a direct factor of $\sigma|_{K^\#_\infty}$. Define the homogeneous vector bundles $\mathscr{V} = \check{E}_\sigma$ and $\mathscr{V}^\# = \check{E}_{\sigma^\#}$, and assume that $\mathscr{V}$, $\mathscr{V}^\#$, and the homomorphism $\mathscr{V}|_{\check{M}(G^\#, X^\#)} \to \mathscr{V}^\#$ are all defined over the extension $k^\#$ of $E(G^\#, X^\#)$. Further assume the following:*

(a) *The representation $\pi^\#$ belongs to the integrable discrete series.*

(b) $\dim H^q(\mathfrak{P}_h, K_\infty, V_0\otimes V_\sigma) = \dim H^q(\mathfrak{P}^\#_h, K^\#_\infty, V^\#_0\otimes V_{\sigma^\#}) = 1$, *and the orthogonal projection $p\otimes p_{\sigma,\sigma^\#} : V\otimes V_\sigma \to V(\pi^\#)\otimes V_{\sigma^\#}$ induces a nontrivial homomorphism*

$$H^q(\mathfrak{P}_h, K_\infty, V_0\otimes V_\sigma) \to H^q(\mathfrak{P}^\#_h, K^\#_\infty, V(\pi^\#)_0\otimes V_{\sigma^\#}),$$

where $V(\pi^\#)$ is the $\pi^\#$-isotypic subspace of V.

(c) *The representation* $\sigma^{\#}$ *satisfies condition* (a) *of Corollary* (5.3.2).

(d) *The image under* cl *of* $H^q(\pi\otimes\sigma)\otimes\mathscr{A}_0(G,\pi)\cong\mathscr{A}_0(G,\pi)\subset\mathscr{H}^q_{\mathrm{cusp},\sigma}$ *is a* $k^{\#}$*-rational subspace of* $\bar{H}^q([\mathscr{V}])$.

Let $\varphi\in\mathscr{H}^q_{\mathrm{cusp},\sigma}$, *and let* $F=\mathrm{cl}(\varphi)\in\bar{H}^q([\mathscr{V}])$. *Then* F *is rational over the extension* L *of* $k^{\#}$ *if and only if, for every* $\gamma\in G(\mathbf{A}^f)$, *the element* $\psi_\gamma(F)$ *is an* L*-rational element of* $\bar{H}^q([\mathscr{V}^{\#}])$.

(7.6.1) Remark. If π and $\pi^{\#}$ are both discrete series representations with sufficiently regular parameter, then it follows from Corollary (5.3.3) that (a), (c), and (d) are automatically satisfied, and σ, $\sigma^{\#}$, and q are uniquely determined by Theorem (4.6.2). Alternatively, one can start with highly regular σ and $\sigma^{\#}$ with highest weights Λ and $\Lambda^{\#}$, respectively, such that $\sigma^{\#}$ is a direct factor of $\sigma|_{K^{\#}_\infty}$ and $q_{\Lambda+\rho}=q_{\Lambda^{\#}+\rho^{\#}}$. Then π and $\pi^{\#}$ are uniquely determined by Lemma (4.7), and the dimensions of the cohomology spaces are as in (4.7)(iii).

Proof. The necessity was discussed above. To prove sufficiency, let $\tau\in\mathrm{Aut}(\mathbb{C}/L)$. We are given that $\psi_\gamma(F)^\tau-\psi_\gamma(F)=\psi_\gamma(F^\tau-F)=0$ for every $\gamma\in G(\mathbf{A}^f)$; we must prove $F^\tau-F=0$. Now hypothesis (d) implies that $F^\tau-F=\mathrm{cl}(\varphi_\tau)$ for some $\varphi_\tau\in H^q(\pi\otimes\sigma)\otimes\mathscr{A}_0(G,\pi)\cong\mathscr{A}_0(G,\pi)$. On the other hand, by Lemma (7.5.5), we have $\mathrm{cl}\circ\alpha_\gamma(\varphi_\tau)=0$ for all $\tau\in\mathrm{Aut}(\mathbb{C}/L)$, $\gamma\in G(\mathbf{A}^f)$. Since cl is injective (Proposition (3.6)), we have

$$\alpha_\gamma(\varphi_\tau)=0\quad\forall\tau\in\mathrm{Aut}(\mathbb{C}/L),\ \gamma\in G(\mathbf{A}^f).$$

By hypothesis (b), this shows that

$$\beta_{\gamma,U_0}(\varphi_\tau)=0\quad\forall\tau\in\mathrm{Aut}(\mathbb{C}/L),\ \gamma\in G(\mathbf{A}^f).\tag{7.6.2}$$

But by Theorem (7.4), (7.6.2) implies $\varphi_\tau=0\ \forall\tau\in\mathrm{Aut}(\mathbb{C}/L)$.

(7.7) At finite level K, this criterion may be checked explicitly. Define the automorphic vector bundle $[\mathscr{V}^{\#}]'$ as in Corollary (2.3), so that the cohomology spaces of $[\mathscr{V}^{\#}]^{\mathrm{can}}$ and $[\mathscr{V}^{\#}]'^{,\mathrm{sub}}$ (resp. $[\mathscr{V}^{\#}]^{\mathrm{sub}}$ and $[\mathscr{V}^{\#}]'^{,\mathrm{can}}$) are placed in duality by Serre duality. Taking into account (3.8.4) and (3.8.5), we thus have an explicit criterion for rationality of cusp forms of type π in terms of values of integrals:

(7.7.1) Corollary. *We retain the notation and hypotheses* (a)–(d) *of Theorem* (7.6). *Let* σ' *be the representation of* $K^{\#}_\infty$ *corresponding to the automorphic vector bundle* $[\mathscr{V}^{\#}]'$; *let* $m=\dim X^{\#}$. *Let* $\varphi\in\mathscr{H}^q_{\mathrm{cusp},\sigma}$, $F=\mathrm{cl}(\varphi)\in\bar{H}^q([\mathscr{V}])$. *Then* F *is rational over the extension* L *of* $k^{\#}$ *if and only if, for every* $\gamma\in G(\mathbf{A}^f)$ *and every cusp form* $\psi\in{}_{K^{\#}}\mathscr{H}^{m-q}_{\mathrm{cusp},\sigma'}$ *such* *that* $\mathrm{cl}(\psi) \in \bar{H}^{m-q}([\mathscr{V}^{\#}]')$ *is L-rational, we have*

$$(7.7.2) \qquad (2\pi i)^{-m} \int_{Z_{G^{\#}}(\mathbf{A})G^{\#}(\mathbb{Q})\backslash G^{\#}(\mathbf{A})} F(h\gamma)\psi(h)\,dh \in L.$$

(7.7.3) Remark. It is not necessary to twist the integral by ξ_{χ}^{-1}, since the central characters of $F(h\gamma)$ and ψ are already inverse to each other.

8. Examples: Forms of GL(2)

(**8.0**) We work out the theory of §7 when $G^{\#}$ is the multiplicative group of a quaternion algebra $B^{\#}$ over $\mathbb{Q}$ which splits over $\mathbb{R}$, and when G is the multiplicative group of $B = R_{F/\mathbb{Q}}B^{\#} \otimes_F \mathbb{Q}$, where F is either $\mathbb{Q}\otimes\mathbb{Q}$ or a real quadratic field. Here $X^{\#}$ is the union of the upper and lower half-planes in $\mathbb{C}$ and $X = X^{\#}\times X^{\#}$. Then $(G^{\#}, X^{\#})$ and (G, X) are naturally basic pairs [46; 17, 2.3], and we have an inclusion $(G^{\#}, X^{\#}) \subset (G, X)$ given by extension of scalars, which reduces to the diagonal map on real points. In this case, as the author learned from Blasius, integrals like (7.7.2) have already been considered by Shimura; we provide a cohomological interpretation of his results in (8.7), using a theorem of Repka [40] on tensor products of discrete series representations.

A forthcoming joint paper with Kudla will investigate this case further, as well as more interesting cases in which $G = \mathrm{GSp}(2, \mathbb{Q})$.

(**8.1**) In (8.1) and (8.2), all unitary representations will be assumed irreducible. We begin by listing the unitary $(\mathfrak{g}^{\#}, K_{\infty}^{\#})$-modules, where $G^{\#}(\mathbb{R}) \cong \mathrm{GL}(2, \mathbb{R})$, with $\bar\partial$-cohomology. In this case, the homomorphism h of (2.0) is an isomorphism $\underline{S} \xrightarrow{\sim} K_{\infty}^{\#}$; we have $\mathfrak{h}_{\mathbb{C}}^{\#} = \mathfrak{k}_{\infty,\mathbb{C}}^{\#} \cong \mathbb{C}\oplus\mathbb{C}$. The elements of $(\mathfrak{h}_{\mathbb{C}}^{\#})^*$ are thus parametrized by ordered pairs $(a, b) \in \mathbb{C}\oplus\mathbb{C}$, normalized by requiring that the character on $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) be $\alpha = (-1, 1)$ (resp. $-\alpha = (1, -1)$), and that the weights of the standard two-dimensional representation of $\mathfrak{g}_{\mathbb{C}}^{\#}$ be $(-1, 0)$ and $(0, -1)$. Then $R^+ = R_n^+ = \{\alpha\}$, $\rho = \frac{1}{2}(-1, 1)$ and, in the notation of (4.6), $\mathscr{F}$ (resp. $\mathscr{F} + \rho$) is the set of ordered pairs $(a, b) \in \mathbb{Z}\oplus\mathbb{Z}$ (resp. $(a, b) \in (\frac{1}{2}+\mathbb{Z})\oplus(\frac{1}{2}+\mathbb{Z})$).

Let (π, V) be a unitary $(\mathfrak{g}^{\#}, K_{\infty}^{\#})$-module. As in (4.4), we are only assuming that π integrates to a unitary representation of $G^{\#,\mathrm{der}}(\mathbb{R}) = \mathrm{SL}(2, \mathbb{R})$. Since $Z_{G^{\#}}(\mathbb{R}) \subset K_{\infty}^{\#}$, π extends to a continuous representation of $Z_{G^{\#}}(\mathbb{R})\cdot G^{\#,\mathrm{der}}(\mathbb{R}) = G^{\#}(\mathbb{R})^0$. The list of unitary representations of $\mathrm{SL}(2, \mathbb{R})$ is well known (cf. e.g. [31]), and it is easy to see that the only unitary $(\mathfrak{g}^{\#}, K_{\infty}^{\#})$-modules with $\bar{\partial}$-cohomology are those whose restriction to $\mathfrak{sl}(2, \mathbb{R})$ are either discrete series, limits of discrete series, or the trivial representation.

(**8.1.1**) Table 1 lists unitary $(\mathfrak{g}^{\#}, K_{\infty}^{\#})$-modules π whose contragredient $\pi^{\#}$ has $\bar{\partial}$-cohomology, with coefficients in the representation σ_{Λ} of $K_{\infty}^{\#}$ given by the character Λ. In the table, λ is the Harish-Chandra parameter of π when π is a discrete series or limit of discrete series, Λ is the given character of $K_{\infty}^{\#}$, ψ is a root with respect to which λ is dominant, q is the dimension in which cohomology occurs, and η is the lowest $K_{\infty}^{\#}$-type occurring in π. In the notation of (4.6), $q = q_{\lambda, \psi}$ and $\eta = \lambda - \rho + \sum_{\alpha \in Q_{\lambda, \psi}} \alpha$.

λ	ψ	π	Λ	q	η
$(a-\frac{1}{2}, b+\frac{1}{2})$, $a>b+1$, $a, b \in \mathbb{Z}$	$\{-\alpha\}$	$\pi = \pi_{\lambda}$ discrete series	(a, b)	0	(a, b)
$(a-\frac{1}{2}, b+\frac{1}{2})$, $a=b+1$, $a, b \in \mathbb{Z}$	$\{-\alpha\}$	$\pi = \pi_{\lambda, \psi}$ limit of disc. series	$(a, b) = (a, a-1)$	0	$(a, b) = (a, a-1)$
$(a-\frac{1}{2}, b+\frac{1}{2})$, $a<b+1$, $a, b \in \mathbb{Z}$	$\{\alpha\}$	$\pi = \pi_{\lambda}$ discrete series	(a, b)	1	$(a-1, b+1)$
$(a-\frac{1}{2}, b+\frac{1}{2})$, $a=b+1$, $a, b \in \mathbb{Z}$	$\{\alpha\}$	$\pi = \pi_{\lambda, \psi}$ limit of disc. series	$(a, b) = (a, a-1)$	1	$(a-1, b+1)$ $= (a-1, a)$
——	-	$\pi = \det^{-a}$	(a, a)	0	(a, a)
——	-	$\pi = \det^{-a}$	$(a+1, a-1)$	1	(a, a)

TABLE 1

(**8.1.2**) The following facts can be read off directly from Table 1:

(**8.1.2.1**) For each pair (Λ, q), there is at most one representation π such that π has $\bar{\partial}$-cohomology in dimension q with coefficients in σ_{Λ}.

(**8.1.2.2**) For Λ of the form (a, b), with $|a - b - 1| > 2$, there is exactly one q such that there exists $\pi = \pi(\Lambda, q)$ with $\bar{\partial}$-cohomology in dimension q with coefficients in σ_{Λ}. This π is the discrete series representation $(\pi_{\Lambda+\rho})^{*}$, and in particular is an *integrable* discrete series representation [29].

(**8.1.2.3**) For Λ of the form (a, b), with $|a - b| < 2$, there is a representation $\pi = \pi(\Lambda, q)$ with $\bar{\partial}$-cohomology in dimension q with coefficients in σ_{Λ} for $q = 0, 1$.

(**8.1.2.4**) Whenever $\pi(\Lambda, q)$ exists, $\dim H^{q}(\mathfrak{P}_{h}^{\#}, K_{\infty}^{\#}, \pi(\Lambda, q) \otimes \sigma_{\Lambda}) = 1$.

(**8.2**) We now study $G(\mathbb{R}) = \mathrm{GL}(2, \mathbb{R}) \times \mathrm{GL}(2, \mathbb{R})$. Any (irreducible) unitary $(\mathfrak{g}, K_\infty)$-module (π, V) is of the form $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$, where (π_i, V_i) is a unitary $(\mathfrak{g}^\#, K_\infty^\#)$-module, $i = 1, 2$. Similarly, any irreducible representation (σ, V_σ) of K_∞ is of the form $(\sigma_1 \otimes \sigma_2, V_{\sigma_1} \otimes V_{\sigma_2})$, where (σ_i, V_{σ_i}) is an irreducible representation of $K_\infty^\#$; in particular, σ is one-dimensional. Such a representation is determined by its highest weight $(\Lambda_1, \Lambda_2) = ((a_1, b_1), (a_2, b_2))$, where $(a_i, b_i) \in \mathbb{Z} \oplus \mathbb{Z}$ is the highest weight of σ_i, in the parametrization of (8.1).

The classification of unitary $(\mathfrak{g}, K_\infty)$-modules with $\bar\partial$-cohomology is thus reduced to table (8.1.1) by the Künneth formula:

$$H^q(\mathfrak{P}_1 \oplus \mathfrak{P}_2, K_1 \times K_2, V_1 \otimes V_2) \cong \bigoplus_{a+b=q} H^a(\mathfrak{P}_1, K_1, V_1) \otimes H^b(\mathfrak{P}_2, K_2, V_2) \tag{8.2.1}$$

[12, 1.3]. The results are listed in Proposition (8.2.2), below. We define a $(\mathfrak{g}, K_\infty)$-module to be *impossible* if it is of the form $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$, where $\pi_1 \not\cong \pi_2$ and at least one of $\{V_1, V_2\}$ is one-dimensional; such modules play no role in the global theory. Also, when λ is a regular character of $K_\infty^\#$, we write $\pi_{\lambda, \psi} = \pi_\lambda$, where ψ is the root of $\mathfrak{g}^\#$ such that $\langle \lambda, \psi \rangle > 0$.

(8.2.2) Proposition. *Let σ_Λ be the irreducible representation of K_∞ given by the character $\Lambda = ((a_1, b_1), (a_2, b_2)) \in (\mathbb{Z} \oplus \mathbb{Z})^2$; let $\lambda_i = (a_i - \frac{1}{2}, b_i + \frac{1}{2})$, $i = 1, 2$. A complete list of unitary $(\mathfrak{g}, K_\infty)$-modules π which are not impossible and such that π^* has $\bar\partial$-cohomology in degree q with coefficients in σ_Λ is given below:*

(i) *If $q = 0$, then either*

(a) $a_1 \geq b_1 + 1$ *and* $a_2 \geq b_2 + 1$, *and* $\pi \cong \pi_{\lambda_1, \psi} \otimes \pi_{\lambda_2, \psi}$, $\psi = \{-\alpha\}$; *or*

(b) $a_1 = b_1 = a_2 = b_2 = a$ *for some* a, *and* $\pi \cong \det^{-a} \otimes \det^{-a}$.

(ii) *If $q = 2$, then either*

(a) $a_1 \leq b_1 + 1$ *and* $a_2 \leq b_2 + 1$, *and* $\pi \cong \pi_{\lambda_1, \psi} \otimes \pi_{\lambda_2, \psi}$, $\psi = \{\alpha\}$; *or*

(b) $a_1 - 1 = b_1 + 1 = a_2 - 1 = b_2 + 1 = a$ *for some* a, *and* $\pi \cong \det^{-a} \otimes \det^{-a}$.

(iii) *If $q = 1$, then $\pi \cong \pi_1 \otimes \pi_2$, where either*

(a) $a_1 \leq b_1 + 1$, $a_2 \geq b_2 + 1$, $\psi_1 = \{\alpha\}$, $\psi_2 = \{-\alpha\}$ *and* $\pi_i = \pi_{\lambda_i, \psi_i}$; *or*

(b) *same as* (a) *with the roles of* π_1 *and* π_2 *switched; or*

(c) $\pi \cong \det^{-a} \otimes \det^{-a}$, *and either* $\Lambda = ((a, a), (a+1, a-1))$ *or* $\Lambda = ((a+1, a-1), (a, a))$.

In particular, the only pairs (Λ, q) for which there exists more than one π which is not impossible and such that π^* has $\bar{\partial}$-cohomology in dimension q with coefficients in σ_Λ are the pairs $\Lambda = ((a, a), (a+1, a-1))$, $q = 1$ and $\Lambda = ((a+1, a-1), (a, a))$, $q = 1$. In that case, there are exactly two such π, of which one is infinite-dimensional and the other one-dimensional. Finally, for any $(\mathfrak{g}, K_\infty)$-modules π which is not impossible, $\dim H^q(\mathfrak{P}_h, K_\infty, \pi^* \otimes \sigma_\Lambda) \leq 1 \ \forall q \geq 0$.

(8.3) Let $B^\#$, B, $X^\#$ and X, be as in (8.0), and let $G^\# = B^{\#,\times}$, $G = B^\times$; define the basic pairs $(G^\#, X^\#) \subset (G, X)$ as in [17]. Note that (G, X) does not satisfy hypothesis (1.1.3), so that the automorphic vector bundles $[\mathscr{V}]$ are only defined in the sense of *stacks* as in [23]. Readers who look askance at stacks should replace G by the subgroup $G^{\mathrm{I}} = \{g \in G | Ng \in \mathbb{G}_{m,\mathbb{Q}}\}$, where N is the reduced norm and $\mathbb{G}_{m,\mathbb{Q}}$ is viewed as an algebraic subgroup of $G^{\mathrm{ab}} = R_{F/\mathbb{Q}}\mathbb{G}_{m,\mathbb{Q}}$. All the results of this paragraph are true for the basic pair (G^1, X), although the parameters have to be changed slightly.

The reflex fields $E(G^\#, X^\#)$ and $E(G, X)$ both coincide with $\mathbb{Q}$ [46, Proposition 9]. Thus $\check{M}^\# = \check{M}(G^\#, X^\#)$ and $\check{M} = \check{M}(G, X) = R_{F/\mathbb{Q}}\check{M}^\# \otimes_F \mathbb{Q}$ are both defined over $\mathbb{Q}$ as homogeneous spaces for $G^\#$ and G, respectively. Now $\check{M}^\#(\mathbb{C}) \cong \mathbb{P}^1(\mathbb{C})$ and $\check{M}(\mathbb{C}) \cong \mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$. Hence $\check{M}^\#$ and $\check{M}$ are clearly the Brauer-Severi varieties, in the sense of [45], associated to the quaternion algebras $B^\#$ and B, respectively.

Any character $\Lambda^\#$ of $K_\infty^\#$ (resp. Λ of K_∞) defines a homogeneous vector bundle $\mathscr{V}_{\Lambda^\#}$ (resp. $\mathscr{V}_\Lambda$) over $\check{M}^\#(\mathbb{C})$ (resp. $\check{M}(\mathbb{C})$). Write $\Lambda^\# = (a, b)$, $\Lambda = ((a_1, b_1), (a_2, b_2))$, as above. Let $\mathscr{O}(d)$ be the canonical vector bundle of degree d on $\mathbb{P}^1$. If we ignore the group action, the vector bundle $\mathscr{V}_{\Lambda^\#}$ (resp. $\mathscr{V}_\Lambda$) is isomorphic to $\mathscr{O}(d(\Lambda^\#))$ (resp. to the exterior tensor product $\mathscr{O}(d_1(\Lambda)) \otimes \mathscr{O}(d_2(\Lambda)))$, where $d(\Lambda^\#) = a - b$ (resp. $d_i(\Lambda) = a_i - b_i$, $i = 1, 2$). On the other hand, the center $Z_{G^\#} \cong \mathbb{Q}^\times$ (resp. $Z_G \cong R_{F/\mathbb{Q}}F^\times$) acts on $\mathscr{V}_{\Lambda^\#}$ (resp. $\mathscr{V}_\Lambda$) by the character $t \mapsto t^{a+b}$ (resp. $t \mapsto \tau_1(t)^{a_1+b_1} \cdot \tau_2(t)^{a_2+b_2}$, where τ_1 and τ_2 are the two homomorphisms from F to $\mathbb{R}$). Now $\mathscr{V}_{\Lambda^\#}$ and $\mathscr{V}_\Lambda$ are determined uniquely as homogeneous vector bundles by the degrees and the central characters. It follows easily from Hilbert's Theorem 90 that

(8.3.1) Lemma. *The homogeneous vector bundle* $\mathscr{V}_{\Lambda^\#}$ (*resp.* $\mathscr{V}_\Lambda$) *is defined over* $\mathbb{C}$ (*resp. over* F; *over* $\mathbb{Q}$ *if* $a_1 = a_2$, $b_1 = b_2$).

As an immediate corollary, we have

(8.4) Theorem. *Let* $(\pi^{\#}, V^{\#})$ *(resp.* (π, V)*) be a unitary* $(\mathfrak{g}^{\#}, K_{\infty}^{\#})$*-module (resp. a unitary* $(\mathfrak{g}, K_{\infty})$*-module). Let* $\Lambda^{\#}$ *(resp.* Λ*) be as in the preceding discussion, and let* $V_{\Lambda^{\#}}$ *(resp.* V_{Λ}*) be the space of the corresponding representation of* $K_{\infty}^{\#}$ *(resp.* K_{∞}*). For any integer* $q \geq 0$, *the image of the homomorphism*

$$H^q(\mathfrak{P}_h^{\#}, K_{\infty}^{\#}, V^{\#} \otimes V_{\Lambda^{\#}}) \otimes \mathrm{Hom}_{(\mathfrak{g}^{\#}, K_{\infty}^{\#})}(V^{\#}, \mathscr{A}_0(G^{\#})) \to \bar{H}^q([\mathscr{V}_{\Lambda^{\#}}])$$

(resp. $H^q(\mathfrak{P}_h, K_{\infty}, V \otimes V_{\Lambda}) \otimes \mathrm{Hom}_{(\mathfrak{g}, K_{\infty})}(V, \mathscr{A}_0(G)) \to \bar{H}^q([\mathscr{V}_{\Lambda}])$*) is rational over* $\mathbb{Q}$ *(resp. over* F*; over* $\mathbb{Q}$ *if* $a_1 = a_2$, $b_1 = b_2$*).*

(8.4.1) Remark. We recall that $\bar{H}^q([\mathscr{V}_{\Lambda}])$ in this theorem, and in Theorem (8.6) below, is only defined in the sense of stacks; the analogous theorem remains true when G is replaced by G^1, defined above.

Proof. By Lemma (8.3.1) and Proposition (5.4.2) we need only consider the case of G when $q = 1$. We first observe the following

(8.4.2) If (π, V) is an impossible $(\mathfrak{g}, K_{\infty})$-module, then

$$\mathrm{Hom}_{(\mathfrak{g}, K_{\infty})}(V, \mathscr{A}(G)) = \{0\}.$$

Indeed, let $\rho = \bigotimes_v \rho_v$ be an irreducible automorphic representation of $G(\mathbf{A})$, where v runs through the places of F. If ρ_v is one-dimensional at some place v at which G is split, then it follows from strong approximation for the simply connected group G^{der} that ρ_v is one-dimensional for all v. Thus if $\pi = \pi_1 \otimes \pi_2$ occurs as the archimedean component of ρ, and one of the π_i's is one-dimensional, then π_1 and π_2 are both one-dimensional. In this case, ρ factors through $G(\mathbf{A})^{\mathrm{ab}} \cong \mathbf{A}_F^{\times}$. Such a ρ is thus given by a Grössencharacter of the totally real field F, from which it follows necessarily that $\pi_1 \cong \pi_2$.

We are thus left with the cases treated in Proposition (8.2.2)(iii). If Λ is not of the form $((a, a), (a+1, a-1))$ or $((a+1, a-1), (a, a))$, then there is only one unitary $(\mathfrak{g}, K_{\infty})$-module $\pi(\Lambda, 1)$, which is not impossible, has $\bar\partial$-cohomology in dimension 1 with coefficients in σ_{Λ}, and is evidently tempered. In this case, our assertion follows from Corollary (5.3.2), Lemma (8.3.1), and Lemma (5.2.3).

Thus, suppose Λ is of the form $((a, a), (a+1, a-1))$ or $((a+1, a-1), (a, a))$. By Theorem (5.3), every element of $\bar{H}^1([\mathscr{V}_{\Lambda}])$ can be represented by a class in $\mathscr{H}^1_{(2), \sigma_{\Lambda}}$. By Proposition (8.2.2) and (8.4.2), exactly two unitary $(\mathfrak{g}, K_{\infty})$-modules π and π' can contribute to $\mathscr{H}^1_{(2), \sigma_{\Lambda}}$.

Of these, one of them, say π, is infinite-dimensional, and π' is one-dimensional. We write

$$\mathscr{H}^1_{(2),\sigma_\Lambda} = \mathscr{H}(\pi) \oplus \mathscr{H}(\pi').$$

It follows from the preceding remarks that $\mathscr{H}(\pi)$ (resp. $\mathscr{H}(\pi')$) is a direct sum of infinite-dimensional (resp. one-dimensional) irreducible $G(\mathbf{A}^f)$-modules. Since the action of $G(\mathbf{A}^f)$ on $\bar{H}^1([\mathscr{V}_\Lambda])$ is rational over the field of definition F of $\mathscr{V}_\Lambda$ (Proposition (2.8)), it follows that the image of $\mathscr{H}(\pi)$ in $\bar{H}^1([\mathscr{V}_\Lambda])$ is rational over F. On the other hand, π is tempered, thus by Wallach's Theorem (5.3.1), $\mathscr{H}(\pi) \subset \mathscr{H}^1_{\text{cusp},\sigma_\Lambda}$. Since $\mathscr{H}(\pi')$ consists of characters of $G(\mathbf{A})$, it even follows that $\mathscr{H}(\pi) = \mathscr{H}^1_{\text{cusp},\sigma_\Lambda}$. This implies the theorem.

(8.4.3) Remark. A similar argument implies the analogous theorem, when F is replaced by an arbitrary totally real commutative semisimple $\mathbb{Q}$-algebra.

(**8.5**) The work of Repka [40] describes the restriction to the diagonal of completed tensor products of discrete series representations of the form $\pi_\lambda \hat{\otimes} \pi_{\lambda'}$, where $q_\lambda = 0$, $q_{\lambda'} = 1$. When $\lambda = (a-\frac{1}{2}, b+\frac{1}{2})$, we write $\pi(a, b)$ for the Hilbert space representation π_λ and $\pi(a, b)_0$ for the associated $(\mathfrak{g}^\#, K^\#_\infty)$-module.

(8.5.1) Theorem (*Repka* [40, *Theorem* **7.3**]). *With the above notation, let* $(a_1, b_1), (a_2, b_2) \in \mathbb{Z}^2$, *with* $a_1 > b_1 + 1$, $a_2 < b_2 + 1$. *Then the restriction to the diagonal* $G^\#(\mathbb{R})^0 \subset G(\mathbb{R})^0$ *of the completed tensor product* $\pi(a_1, b_1)\hat{\otimes}\pi(a_2, b_2)$ *contains as a closed direct summand the representation* $\pi(a', b')$, *with multiplicity one, for every* $(a', b') \in \mathbb{Z}^2$ *such that* (i) $a' + b' = a_1 + b_1 + a_2 + b_2$; (ii) $0 \le |a' - b'| \le |a_1 - b_1 + a_2 - b_2|$; *and* (iii) $a' - b'$ *is of the same sign and parity as* $a_1 - b_1 + a_2 - b_2$.

Proof. Repka's theorem is exactly the above assertion for the restriction of $\pi(a_1, b_1)\hat{\otimes}\pi(a_2, b_2)$ to $G^{\#,\text{der}}(\mathbb{R}) \subset G^\#(\mathbb{R})^0$. It thus remains to verify that the characters of $Z_{G^\#}(\mathbb{R})$ match up; this is just condition (i).

(**8.5.2**) We wish to apply Theorems (8.4) and (8.5.1) to verify the hypotheses of Theorem (7.6) in some cases when $q = 1$, and thus to provide criteria for rationality of cusp forms of the appropriate type. We introduce the following notation:

$$\Lambda = ((a_1, b_1), (a_2, b_2)), \qquad \Lambda^\# = (a_1 + a_2, b_1 + b_2);$$

$$\pi = (\pi(a_1, b_1)\hat{\otimes}\pi(a_2, b_2))^*, \qquad \pi^\# = \pi(a_1 + a_2, b_1 + b_2)^*.$$

We assume that either $a_1 > b_1+1$, $a_2 < b_2+1$ or $a_1 < b_1+1$, $a_2 > b_2+1$, and that moreover $a_1 + a_2 < b_1 + b_2 + 1$. Then π (resp. $\pi^\#$) has $\bar{\partial}$-cohomology in degree 1 with coefficients in σ_Λ (resp. in $\sigma_{\Lambda^\#}$).

Evidently, $\sigma_{\Lambda^\#}$ is the restriction to $K_\infty^\#$ of σ_Λ. Moreover, it follows from Theorem (8.5.1) that $\pi^\#$ occurs with multiplicity one as a closed direct summand of the restriction of π to $G^\#(\mathbb{R})^0$. The orthogonal projection $p: \pi \otimes \sigma_\Lambda \to \pi^\# \otimes \sigma_{\Lambda^\#}$ thus defines a homomorphism

$$(8.5.3) \qquad H(p): H^1(\mathfrak{P}_h, K_\infty, \pi \otimes \sigma_\Lambda) \to H^1(\mathfrak{P}_h^\#, K_\infty^\#, \pi^\# \otimes \sigma_{\Lambda^\#}).$$

We have seen that both spaces in (8.5.3) are one-dimensional.

(8.5.4) Lemma. *The homomorphism* $H(p)$ *of* (8.5.3) *is nontrivial.*

Proof. The quickest proof is a global one. Let $G^\# = \mathrm{GL}(2)_\mathbb{Q}$, $G = G^\# \times G^\#$, so $F = \mathbb{Q} \oplus \mathbb{Q}$; let $M^\# = M(G^\#, X^\#)$, $M = M(G, X) \cong M^\# \times M^\#$. We treat the case $a_1 > b_1 + 1$, $a_2 < b_2 + 1$; the other case is analogous. Let $[\mathscr{V}]$ (resp. $[\mathscr{V}^\#]$) be the automorphic vector bundle over M (resp. $M^\#$) corresponding to σ_Λ (resp. $\sigma_{\Lambda^\#}$). Proposition (8.2.2) and the hypothesis on $\Lambda^\#$ imply that π is the only unitary $(\mathfrak{g}, K_\infty)$-module with $\bar{\partial}$-cohomology with coefficients in σ_Λ. By (7.5.4), Lemma (5.2.3), and Lemma (7.5.5), it thus suffices to prove that the following:

(**8.5.5**) The natural homomorphism $\tau: \bar{H}^1([\mathscr{V}]) \to \bar{H}^1([\mathscr{V}^\#])$ is nontrivial.

Let $\Lambda_1 = (a_1, b_1)$, $\Lambda_2 = (a_2, b_2)$, and $\Lambda_3 = (-a_1 - a_2, -b_1 - b_2)$. Let $[\mathscr{V}_i]$ be the automorphic vector bundle over $M^\#$ corresponding to Λ_i. The Künneth formula for $[\mathscr{V}]$ over $M \cong M^\# \times M^\#$ defines an isomorphism

$$\tilde{H}^1([\mathscr{V}]^{\mathrm{can}}) \xrightarrow{\sim} \tilde{H}^0([\mathscr{V}_1]^{\mathrm{can}}) \otimes \tilde{H}^1([\mathscr{V}_2]^{\mathrm{can}}) \oplus \tilde{H}^1([\mathscr{V}_1]^{\mathrm{can}}) \otimes \tilde{H}^0([\mathscr{V}_2]^{\mathrm{can}})$$

which is easily seen to restrict to an isomorphism

$$(8.5.6) \qquad \bar{H}^1([\mathscr{V}]) \xrightarrow{\sim} \bar{H}^0([\mathscr{V}_1]) \otimes \bar{H}^1([\mathscr{V}_2]) \oplus \bar{H}^1([\mathscr{V}_1]) \otimes \bar{H}^0([\mathscr{V}_2]).$$

By Corollary (3.8.5), we may identify

$$\bar{H}^1([\mathscr{V}_2]) \cong \bar{H}^0([\mathscr{V}_2]')^*, \qquad \bar{H}^1([\mathscr{V}^\#]) \cong \bar{H}^0([\mathscr{V}^\#]')^*.$$

Let $S_1 = \bar{H}^0([\mathscr{V}_1])$, $S_2 = \bar{H}^0([\mathscr{V}_2]')$, and $S_3 = \bar{H}^0([\mathscr{V}^\#]')$. Let $k_1 = a_1 - b_1$, $k_2 = b_2 - a_2 + 2$, and $k_3 = b_1 + b_2 - (a_1 + a_2) + 2$. Our assumptions on Λ and $\Lambda^\#$ imply that $k_i \geq 2$, $i = 1, 2, 3$. With our conventions, S_i is naturally identified with a space of holomorphic elliptic cusp forms of weight k_i, $i = 1, 2, 3$, and is thus nontrivial (cf. [24]).

We want to show that the subspace $\bar{H}^0([\mathscr{V}_1]) \otimes \bar{H}^1([\mathscr{V}_2])$ of $\bar{H}^1([\mathscr{V}])$ maps nontrivially to $\bar{H}^1([\mathscr{V}^\#]) \cong \bar{H}^0([\mathscr{V}^\#]')^*$. By duality, it suffices to show that the homomorphism $S_1 \otimes S_3 \to S_2$ given by the isomorphism

$$[\mathscr{V}_1] \otimes [\mathscr{V}_2]' \xrightarrow{\sim} [\mathscr{V}^{\#}]'$$

is nontrivial. But in terms of our identification above, S_i is a space of cusp forms of weight $k_i \geq 2$, $k_1 + k_3 = k_2$, and the homomorphism $S_1 \otimes S_3 \to S_2$ is just given by multiplication of cusp forms, thus is clearly nontrivial.

We now return to the general case of $G^{\#} = B^{\#,\times}$, $G = B^{\times}$.

(8.6) Theorem. *Let* Λ *and* $\Lambda^{\#}$ *be as in* (8.5.2), *and suppose* $a_1 + a_2 < b_1 + b_2$. *Let* $[\mathscr{V}]$ (*resp.* $[\mathscr{V}^{\#}]$) *be the automorphic vector bundle over* $M(G, X)$ (*resp.* $M(G^{\#}, X^{\#})$) *corresponding to* σ_{Λ} (*resp.* $\sigma_{\Lambda^{\#}}$) (*cf. Remark* (8.4.1)). *Let* σ' *be the representation of* $K_{\infty}^{\#}$ *corresponding to* $[\mathscr{V}^{\#}]'$, *in the notation of* (2.3). *Let* $\varphi \in \mathscr{H}^1_{\mathrm{cusp},\sigma_{\Lambda}}$, *and* $\Phi = \mathrm{cl}(\varphi) \in \bar{H}^1([\mathscr{V}])$. *Then* Φ *is rational over the extension* L *of* F *if and only if, for every* $\gamma \in G(\mathbf{A}^f)$ *and every cusp form* $\psi \in \mathscr{H}^0_{\mathrm{cusp},\sigma'}$ *such that* $\mathrm{cl}(\psi) \in \bar{H}^0([\mathscr{V}^{\#}]')$ *is L-rational, we have*

$$I(\gamma, \Phi, \psi) = (2\pi i)^{-1} \int_{Z_{G^{\#}}(\mathbf{A}) G^{\#}(\mathbb{Q}) \backslash G^{\#}(\mathbf{A})} \Phi(h\gamma)\psi(h)\, dh \in L.$$

Proof. As remarked in (8.1.2.2), the hypothesis $a_1 + a_2 < b_1 + b_2$ implies that the representation $\pi^{\#}$ introduced in (8.5.2) is an integrable discrete series representation; thus hypothesis (a) of Theorem (7.6) is satisfied. Hypotheses (b), (c) and (d) follow from Lemma (8.5.4), (8.1.2.1), and Theorem (8.4), respectively. The theorem is thus a consequence of Corollary (7.7.1).

(8.7) We now assume $B^{\#} = M(2, \mathbb{Q})$ and F is a real quadratic field. Let $\Lambda = ((a_1, b_1), (a_2, b_2))$, and $k_i = a_i - b_i$, $i = 1, 2$. A section s of $[\mathscr{V}_{\Lambda}]$ then defines a classical Hilbert modular form $f = f_s$ of weight (k_1, k_2). Assume $k_i > 1$, $i = 1, 2$. Assume the section s is rational over $\overline{\mathbb{Q}}$. In classical terms, we are requiring that f have algebraic Fourier coefficients on every connected component of $M(G, X)$. Write $\pi(1) = \pi(a_1, b_1)_0$ and $\pi(2) = \pi(a_2, b_2)_0$.

Let $\eta \in \mathrm{Hom}_{(\mathfrak{g}, K_{\infty})}((\pi(1) \otimes \pi(2))^*, \mathscr{A}_0(G))$ be the homomorphism associated to f. Let $W(\eta)$ be the $G(\mathbb{R})$-submodule of $L_2(G(\mathbb{Q})\backslash G(\mathbf{A}))$ generated by the closure of the image of η. Then the space $W(\eta)_0$ of K_{∞}-finite vectors in $W(\eta)$ breaks up as the direct sum of four $(\mathfrak{g}, K_{\infty})$-modules:

(8.7.1) $W(\eta)_0 \cong (\pi(1) \otimes \pi(2))^* \oplus \pi(1) \otimes \pi(2) \oplus \pi(1) \otimes \pi(2)^* \oplus \pi(1)^* \otimes \pi(2).$

The last two summands define cohomology classes in degree 1, and we would like to know which of these is $\bar{\mathbb{Q}}$-rational.

Let τ_1, τ_2 be the two real imbeddings of F. Let $\gamma_0 \in G(\mathbb{Q})$ be an element of $\mathrm{GL}(2, F)$ such that $\tau_1(\det\gamma_0) > 0$ and $\tau_2(\det\gamma_0) < 0$. Let $f^{\gamma_0}(g) = f(g\gamma_0)$, $g \in G(\mathbf{A})$. Then f^{γ_0} belongs to the fourth summand in (8.7.1), and corresponds classically to a C^∞ function on the product of two upper half-planes, holomorphic in the first variable and antiholomorphic in the second, whose standard (nonholomorphic) Fourier expansion at each cusp has coefficients in $\bar{\mathbb{Q}}$. Corresponding to f^{γ_0} is a nontrivial cohomology class s^{γ_0} in $\bar{H}^1([\mathscr{V}_{\Lambda'}])$, where $\Lambda' = ((a_1, b_1), (1-a_2, -1-b_2))$.

As explained in [26, §3], s^{γ_0} is almost certainly not $\bar{\mathbb{Q}}$-rational; if f is not a Hecke eigenform almost everywhere, then s^{γ_0} is probably not even a scalar multiple of a $\bar{\mathbb{Q}}$-rational cohomology class. In fact, suppose f is the transfer, by the Jacquet-Langlands-Shimizu correspondence, of a holomorphic modular form ζ on a quaternion algebra B' over F which is split at τ_1 and ramified at τ_2. Assume ζ is a Hecke eigenform at almost all primes of F, and is $\bar{\mathbb{Q}}$-rational according to Shimura's criterion [49]. Then a special case of a theorem of Shimura [50, II, 3.7(iv)] states that

(8.7.2)
$$\frac{I(\gamma, f^{\gamma_0}, \psi)}{\langle\zeta, \zeta\rangle} \in \bar{\mathbb{Q}} \quad \forall\gamma \in G(\mathbf{A}^f), \forall\psi \in H^0(M(G^\#, X^\#), [\mathscr{V}_{\Lambda^\#}]')(\bar{\mathbb{Q}}),$$

where $\langle\cdot, \cdot\rangle$ is the Petersson inner product, normalized as in [50], and where $\Lambda^\# = (a_1 + 1 - a_2, b_1 - 1 - b_2)$, and $[\mathscr{V}_{\Lambda^\#}]'$ as in (2.3).

If $(b_1 - 1 - b_2) - (a_1 + 1 - a_2) > 2$, then it follows from Theorem (7.4) that $I(\gamma, f^{\gamma_0}, \psi) \neq 0$ for some choice of γ and ψ. Shimura has previously constructed some examples of nonvanishing [50, I, §9]. Theorem (8.6) and (8.7.2) imply the following.

(8.7.3) Under the above hypotheses, $\mathrm{cl}(f^{\gamma_0}/\langle\zeta, \zeta\rangle) \in \bar{H}^1([\mathscr{V}_{\Lambda'}])(\bar{\mathbb{Q}})$.

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